

# A Number of New Generating Functions with Applications to Statistics

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A DISSERTATION

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS  
FOR THE DEGREE OF DOCTOR OF SCIENCE IN THE  
UNIVERSITY OF MICHIGAN

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EMETERIO ROA







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## INTRODUCTION

During the past few years, the graduation of frequency distributions has received considerably more attention than heretofore. Formerly we had the curve of error as the only means of graduating a frequency table; to-day we have at our disposal a number of generating functions which can be used to graduate a fairly large number of frequency distributions. Of these, Pearson's curves are the best known to the English-speaking mathematicians and biometricians. Starting from purely geometrical considerations, Pearson has deduced the system of frequency curves which now bears his name. Charlier and Edgeworth have made valuable contributions based on reasonable assumptions regarding small errors. Kapteyn and Wicksell have developed what the latter terms "the genetic theory of frequency." Thiele, and more recently, Carver, have given a purely mathematical treatment of the subject, showing that Charlier's Type A curve could be arrived at without making the assumptions which Charlier had to make.

Valuable as are the contributions of these men, much can still be done on the subject of graduation. Oftentimes a frequency distribution is found which can not be graduated satisfactorily with any of the functions at our command. Moreover, some of the existing methods of graduating are so laborious that we wonder if other ways could not be devised whereby a great deal of the labor and trouble we now undergo could be avoided.

The object of this paper is to investigate a number of new generating functions which could be employed in graduating frequency distributions. The results we have obtained are intended merely to supplement the generating functions which have been found by other investigators. The method followed in our investigation is to treat the subject of graduation not as a new science but as a mere branch of applied mathematics.



Thus we are able to employ certain mathematical facts which otherwise would have been of no use to us in our work.

This introduction would not be complete without an acknowledgment of the invaluable help I have received from Professor Harry C. Carver. It was he who suggested the problem, and throughout the writing of this paper, he has given me valuable suggestions and helpful criticism. I wish also to express here my feeling of deep indebtedness to Professor James W. Glover for reading the rough draught and for making certain critical suggestions, and to Mr. Raymond W. Barnard for valuable help in the working out of a number of mathematical formulas and in devising some of the labor-saving schemes which have been made a part of this dissertation.



## CHAPTER ONE

If we select at random a number of frequency distributions and plot their corresponding smoothed graphs, we find that a fairly large percentage of the curves generated will, in general form, approximate in varying degrees the probability curve. Some may be skew one way; some the other way. Some slope down from the maximum point more gradually than the normal curve of error; others more sharply. There may be a number that exhibit geometrical properties other than those possessed by the normal curve of error. They may have more than one maximum point. They may, and they generally do, become absolutely zero at some finite points and beyond to plus or minus infinity, instead of becoming merely asymptotic to the axis of reals. But the most striking point, and the one we are primarily interested in, is that all of these curves as plotted from the observed data appear to satisfy the conditions for a function to be expressible as a Fourier's integral.<sup>1</sup> It seems quite reasonable and perfectly justifiable therefore to start with and base our discussion on the following two assumptions:

1. That every frequency distribution is governed by some functional law which can be expressed mathematically.
2. That this law may be expressed as a Fourier's series or a Fourier's integral.

The first of these assumptions is really axiomatic. It can neither be demonstrated nor proved. But it does not have to be proved. It forces itself into our belief. It is the basis of all investigations undertaken and all experiments that are carried on. We observe the present state or condition of things in the hope that the law bringing about such state or condition may be detected, and future happenings then forecasted. If there were no such law, if the happening of an

<sup>1</sup> See Appendix A.

event were more or less of a fatalistic nature, then all the theories and conclusions of empirical and inductive reasoning would be invalid. The second assumption is empirical. There being no way to determine beforehand the functional law of distribution, there can be no direct way of examining its properties. We have to observe and generalize. We have to examine representative cases and from these make empirical deductions.

If absolute accuracy were attainable, the discovery of a functional law would be far easier than it is now. As it is, our tools of observation are so imperfect, and the means of recording impressions so far from being absolutely accurate that every observed frequency table is but a series of approximations into which a multitude of errors enters. With only such approximations instead of the real facts at our command, we can, at best, merely approximate the underlying law of the observed frequency distribution.

Let  $f(x)$  be the functional law of a given distribution. Making use of a well-known form of Fourier's double integral, we have

$$(101) \quad f(x) = \frac{1}{\pi} \int_0^\infty dy \int_{-l_1}^l f(z) \cos y(z-x) dz,$$

where  $l$  and  $l_1$  are the end-points or extremities of the frequency table.

Equation (101) may be reduced to the following more convenient form:

$$(102) \quad f(x) = \frac{1}{2\pi} \int_0^\infty dy \int_{-l_1}^l f(z) \{e^{iy(z-x)} + e^{-iy(z-x)}\} dz;$$

that is

$$(103) \quad f(x) = I_1 + I_2,$$

where

$$(104) \quad I_1 = \frac{1}{2\pi} \int_0^\infty e^{-ixy} \left\{ \int_{-l_1}^l f(z) e^{iyz} dz \right\} dy,$$

and

$$(105) \quad I_2 = \frac{1}{2\pi} \int_0^\infty e^{ixy} \left\{ \int_{-l_1}^l f(z) e^{-iyz} dz \right\} dy.$$



Let us take each one of these integrals and reduce it to a form more convenient for numerical computation. If  $\gamma_1$  and  $\gamma_2$  are two arbitrary parameters which are undetermined for the time being, we may write

$$(106) \quad e^{iyz} = e^{\gamma_1 y i - \gamma_2 (y^2/2)} \cdot e^{iy(z - \gamma_1) + \gamma_2 (y^2/2)},$$

or

$$(107) \quad e^{iyz} = e^{\gamma_1 y i - \gamma_2 (y^2/2)} \cdot \psi(y, z),$$

subject to the condition that

$$(108) \quad \psi(y, z) \equiv e^{iy(z - \gamma_1) + \gamma_2 (y^2/2)}.$$

Expanding  $\psi(y, z)$  by Maclaurin's theorem, we obtain

$$(109) \quad \begin{aligned} \psi(y, z) = 1 + c_1 y i + \frac{c_2}{\underline{2}} y^2 i^2 \\ + \frac{c_3}{\underline{3}} y^3 i^3 + \frac{c_4}{\underline{4}} y^4 i^4 + \dots, \end{aligned}$$

where the successive coefficients  $c_1, c_2, c_3 \dots c_n$  are functions of  $\gamma_1, \gamma_2$  and  $z$ . Whence

$$(110) \quad e^{iyz} = e^{\gamma_1 y i - \gamma_2 (y^2/2)} \left\{ 1 + c_1 y i + \frac{c_2}{\underline{2}} y^2 i^2 + \frac{c_3}{\underline{3}} y^3 i^3 + \dots \right\}.$$

The evaluation of the coefficients is effected by the use of the equation

$$(111) \quad c_n \cdot i^n = D_y^n [e^{iy(z - \gamma_1) + \gamma_2 (y^2/2)}]_{y=0}.$$

From this we get the following values:

$$(112) \quad \begin{aligned} c_1 &= z - \gamma_1, \\ c_2 &= (z - \gamma_1)^2 - \gamma_2, \\ &\cdot \\ &\cdot \\ &\cdot \\ c_n &= \left\{ (z - \gamma_1)^n - \frac{n^{(2)}}{2} \gamma_2 (z - \gamma_1)^{n-2} \right. \\ &\quad \left. + \frac{n^{(4)}}{2^2 \underline{2}} \gamma_2^2 (z - \gamma_1)^{n-4} + \dots \right\}. \end{aligned}$$

The arbitrary constants  $\gamma_1$  and  $\gamma_2$  may now be determined by imposing the conditions that

$$(113) \quad \int_{-l_1}^l c_1 y i f(z) dz = \int_{-l_1}^l \frac{c_2}{2} y^2 i^2 f(z) dz = 0.$$

Substituting the values of  $c_1$  and  $c_2$  just found and solving, we get the following values of  $\gamma_1$  and  $\gamma_2$ :

$$(114) \quad \gamma_1 = \frac{\int_{-l_1}^l z f(z) dz}{\int_{-l_1}^l f(z) dz};$$

$$(115) \quad \gamma_2 = \frac{\int_{-l_1}^l (z - \gamma_1)^2 f(z) dz}{\int_{-l_1}^l f(z) dz}.$$

It must be recalled that  $l$  and  $l_1$  are the end-points or extremities of the frequency distribution while  $f(z)$  is the functional law. Whence the definite integral

$$\int_{-l_1}^l f(z) dz$$

representing the area under the curve  $f(z)$  is equal to the total frequency. Equation (114), expressed in statistical language, is merely a statement that  $\gamma_1$  is the distance of the mean from the arbitrary origin, while (115) says that  $\gamma_2$  is the second moment around the centroid vertical; therefore

$$(116) \quad \begin{aligned} \gamma_1 &= b, \\ \gamma_2 &= \mu_2 = \sigma^2. \end{aligned}$$

From equations (104), (110) and (116) we have

$$(117) \quad I_1 = \frac{N}{2\pi} \int_0^\infty e^{-(x-b)yi - (\sigma^2 y^2/2)} \left\{ 1 + \frac{a_3}{3} y^3 i^3 + \frac{a_4}{4} y^4 i^4 + \dots \right\} dy,$$

where

$$(118) \quad \begin{aligned} N &= \text{the total frequency,} \\ b &= \text{distance of the mean from the arbitrary origin,} \\ \sigma &= \text{the standard deviation,} \end{aligned}$$



and

$$(119) \quad a_n = \frac{\int_{-i_1}^i c_n \cdot f(z) dz}{N}.$$

In a similar manner, we may prove that

$$(120) \quad I_2 = \frac{N}{2\pi} \int_0^\infty e^{(x-b)y i - (\sigma^2 y^2/2)} \left\{ 1 - \frac{a_3}{[3]} y^3 i^3 + \frac{a^4}{[4]} y^4 i^4 - \dots \right\} dy.$$

Using (117) and (120), we get the following expression for the functional law:

$$(121) \quad f(x) = \frac{N}{2\pi} \int_{-\infty}^\infty e^{-(x-b)y i - (\sigma^2 y^2/2)} \left\{ 1 + \frac{a_3}{[3]} y^3 i^3 + \frac{a^4}{[4]} y^4 i^4 + \dots \right\} dy.$$

And, finally, using the well-known definite integral of Laplace

$$(122) \quad \frac{1}{\pi} \int_0^\infty dy e^{-(\sigma^2 y^2/2)} \cos (x-b)y = \frac{1}{\sigma \sqrt{2\pi}} e^{-[(x-b)^2/2\sigma^2]},$$

and remembering that

$$(123) \quad \frac{1}{\pi} \int_0^\infty dy e^{-(\sigma^2 y^2/2)} \cos (x-b)y = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-(x-b)y i - (\sigma^2 y^2/2)} dy,$$

we have

$$(124) \quad f(x) = N \left\{ \varphi(x) - \frac{a_3}{[3]} \varphi^3(x) + \frac{a^4}{[4]} \varphi^4(x) - \dots \right\},$$

where

$$(125) \quad \varphi(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-[(x-b)^2/2\sigma^2]}.$$

It must be observed that (124) is obtained by integrating term by term the series in the right-hand member of equation (121). To justify such a step, the nature of  $f(x)$  and the convergence of the right-hand member of (124) must be investigated. Such investigation would be extremely difficult if the aim is to arrive at general conclusions. The convergence

of the series in the right-hand member of (124) ultimately depends upon the nature of  $f(x)$ , the functional law, which is undetermined and uncertain.

Given an observed frequency distribution, a trial graduation is made by using (124) as a graduating function. This may or may not give a satisfactory fit. It may be that the first two, or three, or four terms of the series will be enough to give a fairly good graduation of the observed data. If such is the case, then we know that the series converges quite rapidly to the true value. Such a graduation would then be accepted both as a final result and a justification of equation (124). On the other hand, if the values of the successive terms of the right-hand member of equation (124) show no tendency to converge rapidly, then (124) should not be used for graduation and hence the step in question need not be justified. Thus the practical application of (124) automatically supplies the justification of the procedure by which it has been obtained.

From (112) and (119) we obtain the following values of the successive  $a$ -coefficients:

$$\begin{aligned}
 a_3 &= \mu_3, \\
 a_4 &= \mu_4 - 3\sigma^4, \\
 a_5 &= \mu_5 - 10\sigma^2\mu_3, \\
 a_6 &= \mu_6 - 15\sigma^2\mu_4 + 30\sigma^6, \\
 (126) \quad a_7 &= \mu_7 - 21\sigma^2\mu_5 + 105\sigma^4\mu_3, \\
 a_8 &= \mu_8 - 28\sigma^2\mu_6 + 210\sigma^4\mu_4 - 315\sigma^8, \\
 &\cdot \\
 &\cdot \\
 &\cdot
 \end{aligned}$$

Equation (124) does not reveal a new generating function. It is quite well known and is commonly referred to as Charlier's Type A curve. Edgeworth, Charlier, Thiele and Carver have contributed a number of different ways of deducing it. The foregoing development has been given to show the possibility of using Fourier's double integral in mathematical statistics.

The successive derivatives of  $\varphi(x)$  may be readily evaluated by the use of a recurring formula. Dropping the constant

coefficient, we have

$$\varphi(x) = e^{-[(x-b)^2/2\sigma^2]},$$

from which we at once obtain the first derivative

$$\varphi'(x) = \frac{b-x}{\sigma^2} \cdot e^{-[(x-b)^2/2\sigma^2]} = \frac{b-x}{\sigma^2} \varphi(x),$$

or rearranging terms, and differentiating,

$$\begin{aligned} \sigma^2 \varphi' - b\varphi &= -x\varphi, \\ \sigma^2 \varphi^2 - b\varphi' &= -x\varphi' - \varphi, \\ \sigma^2 \varphi^3 - b\varphi^2 &= -x\varphi^2 - 2\varphi', \\ \sigma^2 \varphi^4 - b\varphi^3 &= -x\varphi^3 - 3\varphi^2, \\ &\vdots \\ &\vdots \\ &\vdots \\ \sigma^2 \varphi^{n+1} - b\varphi^n &= -x\varphi^n - n\varphi^{n-1}, \end{aligned} \tag{127}$$

whence the following equation:

$$\sigma^2 \varphi^{n+1} + (x-b)\varphi^n + n\varphi^{n-1} = 0. \tag{128}$$

It can be proved by repeated differentiation that in general the ratio of the  $n$ th derivative of  $\varphi(x)$  to the original function is equal to the ratio of an  $n$ th degree polynomial in  $(1/\sigma)(x-b)$  to the  $n$ th power of the standard deviation. That is,

$$\frac{\varphi^n(x)}{\varphi(x)} = \frac{1}{\sigma^n} H_n(t), \tag{129}$$

where

$$t = \frac{1}{\sigma}(x-b)$$

and

$$H_n(t) = \sum_{r=0}^n A_r \cdot t^r.$$

When  $n$  is odd, the coefficients of  $t^0$  and of all the even powers of  $t$  vanish; when  $n$  is even, the coefficients of the odd powers of  $t$  vanish.

If we give to  $n$  the values 0, 1, 2, 3, 4,  $\dots$  successively, the function  $H_n(t)$  gives rise to a set of polynomials known in mathematical literature as Hermite's polynomials, the first



seven of which are

$$\begin{aligned}
 H_0(t) &= 1, \\
 H_1(t) &= t, \\
 H_2(t) &= t^2 - 1, \\
 (130) \quad H_3(t) &= t^3 - 3t, \\
 H_4(t) &= t^4 - 6t^2 + 3, \\
 H_5(t) &= t^5 - 10t^3 + 15t, \\
 H_6(t) &= t^6 - 15t^4 + 45t^2 - 15.
 \end{aligned}$$

Other Hermite's polynomials can be computed very readily since, for all values of  $n$ , we have the recurring relation

$$(131) \quad H_{n+1}(t) + t \cdot H_n(t) + n \cdot H_{n-1}(t) = 0.$$

This equation may be deduced from (128) and (129).

A table giving the numerical values of the Hermite's polynomials to ten decimal places for  $n = 2, 3, 4, 5, 6$  has been published by the Danish actuary, Jørgensen.<sup>1</sup> The range of the argument is from 0.01 to 4.00 inclusive.

If in equation (124) we substitute  $t$  for  $(1/\sigma)(x - b)$ , the series-expansion would take the following form:

$$\begin{aligned}
 f(x) = N \left\{ \varphi(t) - \frac{a_3'}{\underline{3}} \varphi^3(t) + \frac{a_4'}{\underline{4}} \varphi^4(t) \right. \\
 (132) \quad \left. - \frac{a_5'}{\underline{5}} \varphi^5(t) + \dots \right\},
 \end{aligned}$$

where

$$\begin{aligned}
 a_3' &= \alpha_3, \\
 a_4' &= \alpha_4 - 3, \\
 a_5' &= \alpha_5 - 10\alpha_3, \\
 a_6' &= \alpha_6 - 15\alpha_4 + 30, \\
 (133) \quad a_7' &= \alpha_7 - 21\alpha_5 + 105\alpha_3, \\
 a_8' &= \alpha_8 - 28\alpha_6 + 210\alpha_4 - 315, \\
 &\vdots \\
 &\vdots \\
 &\vdots
 \end{aligned}$$

$\alpha_n$  being identical to  $\mu_n/\sigma^n$ .

<sup>1</sup> *Undersøgelser over Frekvensflader og Korrelation*. N. R. Jørgensen. Copenhagen. 1916. •

## FIRST NUMERICAL EXAMPLE

We shall now show in detail a graduation of a frequency table by means of Charlier's Type A curve. The observed distribution is taken from *The Medical Department of the U. S. Army in the World War*, volume XV, Part I, page 522. It shows the heights of 6441 colored soldiers.

The first thing done is to calculate the unadjusted moments of the observed frequency distribution around some arbitrary origin. The first eight moments are evaluated by a direct application of the formula

$$(134) \quad \nu_n' = \frac{\sum x^n f(x)}{\sum f(x)},$$

where  $\nu_n'$  stands for the  $n$ th unadjusted moment of the rough data around the mid-point of the class 172-173. To reduce the values thus found to moments around the mean, the following method is suggested. Calculate the ratios

$$\frac{\nu_n'}{b^n},$$

$b$  being the distance of the mean from the arbitrary origin and is identical therefore to  $\nu_1'$ . These ratios are tabulated in one column and differenced. Multiply the leading differences by the corresponding powers of  $b$  and the products are the required unadjusted moments around the mean. Or, stated algebraically,

$$(135) \quad \nu_s = b^s \cdot \Delta_{n=0}^s \frac{\nu_n'}{b^n}.$$

But, since we are dealing with graduated variates, the moments have to be adjusted. Sheppard's adjustment formulas are used to reduce  $\nu_s$  to  $\mu_s$ , the  $s$ th adjusted moment about the mean. Divide  $\mu_s$  by  $\sigma^s$ , and obtain the value of  $\bar{\alpha}_s$ .

The columns in Table 1 give the different steps in the preliminary work of evaluating the alphas of the observed distribution.

Table 2 gives the process of graduation step by step, while Table 3 gives the observed distribution and the different

TABLE 1

TABLE SHOWING THE SUCCESSIVE STEPS IN CALCULATING THE ALPHAS OF AN OBSERVED DISTRIBUTION

*First Numerical Example*

| <i>s</i> | Moment about Arbitrary Origin | Powers of <i>b</i> , Distance of Mean from Arbitrary Origin | Ratio of Column (2) to Column (3) |
|----------|-------------------------------|-------------------------------------------------------------|-----------------------------------|
|          | $\nu_s'$                      | $b^s$                                                       | $\nu_s'/b^s$                      |
| (1)      | (2)                           | (3)                                                         | (4)                               |
| 1        | -.30414532                    | -.30414532                                                  | 1                                 |
| 2        | 11.733116                     | .092504376                                                  | 126.83850                         |
| 3        | -6.1821146                    | -.028134773                                                 | 219.73217                         |
| 4        | 433.84397                     | .0085570595                                                 | 50700.123                         |
| 5        | -55.050303                    | -.0026025896                                                | 21152.126                         |
| 6        | 27851.63996                   | .00079156545                                                | 35185517                          |
| 7        | 19240.72986                   | -.00024075093                                               | -79919649                         |
| 8        | 2458950.65813                 | .000073223268                                               | 33581548000                       |

*Calculation of Standard Deviation*

$$\nu_2 = \nu_2' - b^2 = 11.640612$$

$$\mu_2 = \nu_2 - \frac{1}{12} = 11.557279$$

$$\sigma = \sqrt{\mu_2} = 3.3995998$$

| <i>s</i> | Unadjusted Moment about Mean                                | Adjusted Moment about Mean | Powers of Standard Deviation | Observed Alpha                            |
|----------|-------------------------------------------------------------|----------------------------|------------------------------|-------------------------------------------|
|          | $\nu_s = b^s \cdot \frac{\Delta^s \nu_n'}{n = 0 \quad b^n}$ | $\mu_s$                    | $\sigma^s$                   | $\bar{\alpha}_s = \frac{\mu_s}{\sigma^s}$ |
| (1)      | (5)                                                         | (6)                        | (7)                          | (8)                                       |
| 3        | 4.4673331                                                   | 4.4673331                  | 39.290123                    | .11370117                                 |
| 4        | 432.80943                                                   | 427.01829                  | 133.57069                    | 3.1969461                                 |
| 5        | 602.27975                                                   | 598.55697                  | 454.08689                    | 1.3181551                                 |
| 6        | 28351.188                                                   | 27815.246                  | 1543.7137                    | 18.018397                                 |
| 7        | 78856.410                                                   | 77806.981                  | 5248.0088                    | 14.826000                                 |
| 8        | 2571330.7                                                   | 2506054.1                  | 17841.130                    | 140.46499                                 |

graduated distributions. These theoretical graduations are obtained by multiplying columns (10), (11), and (12) of Table 2 by 6441, the total frequency.

The values of  $\int_{-\infty}^t \varphi(t)dt$  are from Pearson's Tables for Statisticians and Biometricians.<sup>1</sup> The values in columns (4)

<sup>1</sup> *Tables for Statisticians and Biometricians*. Edited by Karl Pearson. Cambridge University Press. 1914.



and (6) of Table 2 are taken from the mathematical tables of Glover.<sup>1</sup> Straight-line interpolation is employed whenever intermediate values are needed.

It must be observed that the foregoing graduation consists of evaluating the areas under the theoretical curve from  $-\infty$  to the boundary points of the class intervals and then differencing these areas. The very nature of the rough data graduated demands this procedure. Height is essentially a graduated variate; hence tables of heights must be treated as tables of areas rather than of ordinates. Most of the statistics we come across in actual practice are of graduated variates. Height, weight, age, temperature; all these differ by infinitesimals and the curves representing a distribution according to one such characteristic must of necessity be continuous rather than discrete. Hence the advantage of a generating function which can be integrated. It may be argued that, in case the generating function employed is not integrable, quadrature formulas could be used to obtain areas from ordinates. This could be done, but the values found would be, at best, mere approximations, and would therefore be subject to discrepancies.

Table 4 gives the final results of another graduation by means of the generating function we have discussed in this chapter. Here we have gone one step further than in the preceding example. Four theoretical distributions are given, the fourth being graduated by taking four terms of the series-expansion:

$$(136) \quad f(x) = \varphi(t) - \frac{a_3'}{3} \varphi^3(t) + \frac{a_4'}{4} \varphi^4(t) - \frac{a_5'}{5} \varphi^5(t);$$

whereas in the preceding example we have not gone farther than the third term.

<sup>1</sup> *Tables of Applied Mathematics in Finance, Insurance, Statistics.* Edited by James W. Glover. Ann Arbor. 1923.

TABLE 2  
TABLE SHOWING THE SUCCESSIVE STEPS IN GRADUATING A DISTRIBUTION OF HEIGHTS OF 6441 COLORED SOLDIERS.  
GENERATING FUNCTION USED:  $\varphi(x) = \mathbf{K}e^{-(x^2, y^2)}$

| Boundary Point<br>of<br>Class Interval | (1) | (2)                         | Integral of<br>Generating<br>Function | Second<br>Derivative | Second<br>Term              | Third<br>Derivative | Third<br>Term              | Integral of Expansion of |                | Graduated Unit Frequency |                   |                     | Same as<br>Column (1) |
|----------------------------------------|-----|-----------------------------|---------------------------------------|----------------------|-----------------------------|---------------------|----------------------------|--------------------------|----------------|--------------------------|-------------------|---------------------|-----------------------|
|                                        |     |                             |                                       |                      |                             |                     |                            | Two<br>Terms             | Three<br>Terms | One Term<br>Used         | Two<br>Terms Used | Three<br>Terms Used |                       |
| $x$                                    |     | $t = \frac{1}{\sigma}(x-b)$ | $\int_{-\infty}^t \varphi^{(i)} dt$   | $\varphi^{(i)}$      | $-\frac{a_3}{3} \times (4)$ | $\varphi^{(i)}$     | $\frac{a_4}{4} \times (6)$ | (3) + (5)                | (7) + (8)      | $\Delta(3)$              | $\Delta(8)$       | $\Delta(9)$         | $x$                   |
|                                        | (1) |                             | (3)                                   | (4)                  | (5)                         | (6)                 | (7)                        | (8)                      | (9)            | (10)                     | (11)              | (12)                | (1a)                  |
| -15.5                                  |     | -4.470                      | .00000                                | .00035               | -.00001                     | .00139              | .00001                     | -.00001                  | .00000         | .00001                   | .00000            | .00002              | -15.5                 |
| -14.5                                  |     | -4.176                      | .00001                                | .00107               | -.00002                     | .00393              | .00003                     | -.00001                  | .00002         | .00001                   | .00000            | .00005              | -14.5                 |
| -13.5                                  |     | -3.882                      | .00005                                | .00300               | -.00006                     | .00998              | .00008                     | -.00001                  | .00007         | .00004                   | .00004            | .00015              | -13.5                 |
| -12.5                                  |     | -3.587                      | .00017                                | .00761               | -.00014                     | .02269              | .00019                     | .00003                   | .00022         | .00033                   | .00014            | .00032              | -12.5                 |
| -11.5                                  |     | -3.293                      | .00050                                | .01735               | -.00033                     | .04553              | .00037                     | .00017                   | .00054         | .00085                   | .00051            | .00080              | -11.5                 |
| -10.5                                  |     | -2.999                      | .00135                                | .03553               | -.00067                     | .07991              | .00066                     | .00068                   | .00134         | .00207                   | .00151            | .00184              | -10.5                 |
| -9.5                                   |     | -2.705                      | .00342                                | .06495               | -.00123                     | .12006              | .00099                     | .00219                   | .00318         | .00453                   | .00377            | .00399              | -9.5                  |
| -8.5                                   |     | -2.411                      | .00795                                | .10497               | -.00199                     | .14791              | .00121                     | .00596                   | .00717         | .00918                   | .00837            | .00825              | -8.5                  |
| -7.5                                   |     | -2.117                      | .01713                                | .14774               | -.00280                     | .13310              | .00109                     | .01433                   | .01542         | .01702                   | .01649            | .01577              | -7.5                  |
| -6.5                                   |     | -1.823                      | .03415                                | .17594               | -.00333                     | .04462              | .00037                     | .03082                   | .03119         | .02911                   | .02930            | .02789              | -6.5                  |
| -5.5                                   |     | -1.528                      | .06326                                | .16570               | -.00314                     | -.12619             | -.00104                    | .06012                   | .05908         | .04534                   | .04663            | .04488              | -5.5                  |
| -4.5                                   |     | -1.234                      | .10860                                | .09739               | -.00185                     | -.33963             | -.00279                    | .10675                   | .10396         | .06501                   | .06743            | .06603              | -4.5                  |
| -3.5                                   |     | -.940                       | .17361                                | -.02985              | .00057                      | -.51023             | -.00419                    | .17418                   | .16999         | .08553                   | .08854            | .08830              | -3.5                  |
| -2.5                                   |     | -.646                       | .25914                                | -.18868              | .00358                      | -.54022             | -.00443                    | .26272                   | .25829         | .10328                   | .10593            | .10724              | -2.5                  |
| -1.5                                   |     | -.352                       | .36242                                | -.32851              | .00623                      | -.37961             | -.00312                    | .36865                   | .36553         | .11445                   | .11574            | .11829              | -1.5                  |

TABLE 2—(Continued)

| (1)  | (2)   | (3)     | (4)      | (5)      | (6)      | (7)      | (8)     | (9)    | (10)   | (11)   | (12)   | (1a)  |
|------|-------|---------|----------|----------|----------|----------|---------|--------|--------|--------|--------|-------|
| —    | —     | .47687  | — .39692 | .00752   | — .06922 | — .00057 | .48439  | .48382 | .11680 | .11622 | .11901 | — 0.5 |
| 0.5  | .237  | .59367  | — .36610 | .00694   | .27062   | .00222   | .60061  | .60283 | .10862 | .10639 | .10827 | 0.5   |
| 1.5  | .531  | .70229  | — .24779 | .00471   | .50006   | .00410   | .70700  | .71110 | .09302 | .09003 | .09039 | 1.5   |
| 2.5  | .825  | .79531  | — .09066 | .00172   | .54315   | .00446   | .79703  | .80149 | .07312 | .07038 | .06934 | 2.5   |
| 3.5  | 1.119 | .86843  | .05379   | — .00102 | .41719   | .00342   | .86741  | .87083 | .03274 | .05098 | .04927 | 3.5   |
| 4.5  | 1.413 | .92117  | .14650   | — .00278 | .20845   | .00171   | .91839  | .92010 | .03492 | .03433 | .03273 | 4.5   |
| 5.5  | 1.707 | .95609  | .17786   | — .00337 | .01368   | .00011   | .95272  | .95283 | .02121 | .02151 | .02051 | 5.5   |
| 6.5  | 2.001 | .97730  | .16186   | — .00307 | — .10825 | — .00089 | .97423  | .97334 | .01186 | .01262 | .01229 | 6.5   |
| 7.5  | 2.296 | .98916  | .12212   | — .00231 | — .14910 | — .00122 | .98685  | .98563 | .00604 | .00584 | .00596 | 7.5   |
| 8.5  | 2.590 | .99520  | .07957   | — .00151 | — .13388 | — .00110 | .99369  | .99259 | .00284 | .00349 | .00381 | 8.5   |
| 9.5  | 2.884 | .99804  | .04562   | — .00086 | — .09561 | — .00078 | .99718  | .99640 | .00122 | .00164 | .00195 | 9.5   |
| 10.5 | 3.178 | .99926  | .02327   | — .00044 | — .05770 | — .00047 | .99882  | .99835 | .00048 | .00072 | .00094 | 10.5  |
| 11.5 | 3.472 | .99974  | .01064   | — .00020 | — .03025 | — .00025 | .99954  | .99929 | .00018 | .00030 | .00044 | 11.5  |
| 12.5 | 3.766 | .99992  | .00438   | — .00008 | — .01399 | — .00011 | .99984  | .99973 | .00006 | .00011 | .00017 | 12.5  |
| 13.5 | 4.061 | .99998  | .00162   | — .00003 | — .00574 | — .00005 | .99995  | .99990 | .00001 | .00003 | .00006 | 13.5  |
| 14.5 | 4.355 | .99999  | .00055   | — .00001 | — .00211 | — .00002 | .99998  | .99996 | .00001 | .00002 | .00003 | 14.5  |
| 15.5 | 4.649 | 1.00000 | .00017   | — .00000 | — .00070 | — .00001 | 1.00000 | .99999 | .00001 |        |        | 15.5  |



TABLE 3

DISTRIBUTION OF HEIGHTS OF 6441 COLORED SOLDIERS

*First Numerical Example**Source of Rough Data: The Med. Dept. of U. S. Army, vol. XV, Part I, p. 522.**Generating Function Employed:  $\varphi(t) = \kappa e^{-(t^2/2)}$ .*

| Class<br>Interval | Observed<br>Frequency | Graduation       |                   |                     |
|-------------------|-----------------------|------------------|-------------------|---------------------|
|                   |                       | One Term<br>Used | Two Terms<br>Used | Three Terms<br>Used |
|                   |                       | First            | Second            | Third               |
| (1)               | (2)                   | (3)              | (4)               | (5)                 |
| Centimeters       |                       |                  |                   |                     |
| 146-147           | 0                     | 0                | 0                 | 1                   |
| 148-149           | 2                     | 2                | 1                 | 2                   |
| 150-151           | 9                     | 6                | 3                 | 5                   |
| 152-153           | 13                    | 13               | 10                | 12                  |
| 154-155           | 23                    | 29               | 24                | 26                  |
| 156-157           | 56                    | 59               | 54                | 53                  |
| 158-159           | 88                    | 110              | 106               | 102                 |
| 160-161           | 162                   | 188              | 189               | 180                 |
| 162-163           | 318                   | 292              | 300               | 289                 |
| 164-165           | 468                   | 419              | 434               | 425                 |
| 166-167           | 564                   | 551              | 570               | 569                 |
| 168-169           | 665                   | 665              | 682               | 691                 |
| 170-171           | 708                   | 737              | 746               | 762                 |
| 172-173           | 749                   | 752              | 749               | 767                 |
| 174-175           | 747                   | 700              | 685               | 697                 |
| 176-177           | 586                   | 599              | 580               | 582                 |
| 178-179           | 469                   | 471              | 453               | 447                 |
| 180-181           | 314                   | 340              | 328               | 317                 |
| 182-183           | 207                   | 225              | 221               | 211                 |
| 184-185           | 133                   | 137              | 139               | 132                 |
| 186-187           | 70                    | 77               | 81                | 79                  |
| 188-189           | 38                    | 39               | 44                | 45                  |
| 190-191           | 22                    | 18               | 23                | 24                  |
| 192-193           | 15                    | 8                | 11                | 13                  |
| 194-195           | 10                    | 3                | 5                 | 6                   |
| 196-197           | 3                     | 1                | 2                 | 3                   |
| 198-199           | 2                     | 0                | 1                 | 1                   |

TABLE 4

DISTRIBUTION OF ARM LENGTHS OF 5514 COLORED SOLDIERS  
*Second Numerical Example*

*Source of Rough Data: The Med. Dept. of U. S. Army, vol. XV, Part I, p. 531.*

*Generating Function Employed:  $\varphi(t) = ke^{-(t^2/2)}$ .*

| Class Interval | Observed Frequency | Graduation    |                |                  |                 |
|----------------|--------------------|---------------|----------------|------------------|-----------------|
|                |                    | One Term Used | Two Terms Used | Three Terms Used | Four Terms Used |
|                |                    | First         | Second         | Third            | Fourth          |
| (1)            | (2)                | (3)           | (4)            | (5)              | (6)             |
| Centimeters    |                    |               |                |                  |                 |
| 62-63          | 0                  | 1             | 0              | 0                | 0               |
| 64-65          | 0                  | 3             | 0              | 0                | 0               |
| 66-67          | 0                  | 11            | 4              | 4                | 4               |
| 68-69          | 16                 | 31            | 20             | 20               | 21              |
| 70-71          | 88                 | 94            | 82             | 82               | 84              |
| 72-73          | 220                | 205           | 205            | 205              | 206             |
| 74-75          | 399                | 386           | 409            | 409              | 408             |
| 76-77          | 614                | 614           | 657            | 657              | 653             |
| 78-79          | 909                | 819           | 858            | 858              | 854             |
| 80-81          | 921                | 918           | 923            | 924              | 923             |
| 82-83          | 822                | 863           | 832            | 832              | 836             |
| 84-85          | 632                | 681           | 637            | 637              | 641             |
| 86-87          | 441                | 452           | 422            | 422              | 423             |
| 88-89          | 246                | 252           | 246            | 245              | 244             |
| 90-91          | 119                | 118           | 127            | 127              | 125             |
| 92-93          | 52                 | 46            | 58             | 58               | 57              |
| 94-95          | 22                 | 15            | 23             | 23               | 23              |
| 96-97          | 8                  | 4             | 8              | 8                | 8               |
| 98-99          | 5                  | 1             | 3*             | 3*               | 4*              |

\* Includes area beyond the interval 98-99.

## CHAPTER TWO

It is a well-known fact that a graduation by Charlier's Type A curve leads not infrequently to unsatisfactory results. Even excluding those of the extremely asymmetrical type, there is still a large number of frequency distributions for the graduation of which other generating functions must be employed. If, for example, the  $\alpha_4$  of the observed distribution is much greater or considerably smaller than 3, the resulting graduation, if only the first three terms of the expansion are used, is, in general, not satisfactory and oftentimes ridiculous. An improvement might be effected by employing as many as nine or ten terms of the series, but the amount of labor and time such a procedure would entail is prohibitive.

Several attempts have been made to supplement the normal curve of error with other related generating functions which might be used in case Charlier's Type A curve fails to give a satisfactory fit. The so-called transformation of variates may be mentioned as an example.<sup>1</sup>

We shall develop now a generalization of Charlier's Type A curve which may prove useful in graduating those frequency distributions which can not be graduated satisfactorily by the use of the normal curve of error as a generating function. In other words, we shall present a family of generating functions of which the normal curve of error is a member.

From (101) we have the following fundamental equation:

$$(201) \quad f(x) = \frac{1}{\pi} \int_0^\infty dy \int_{-iy}^1 f(z) \cos y(z - x) dz.$$

Let  $f(x)$  now represent the equation of the curve when the mean is arbitrarily chosen as the origin. Now let

$$t = x^q,$$

<sup>1</sup> See *The Mathematical Theory of Probabilities*. Arne Fisher. Chapter XVI.

where  $q$  is a positive odd integer, so that

$$F(t) \equiv f(x).$$

Then, obviously, from (201),

$$\begin{aligned} f(x) \equiv F(t) &= \frac{1}{\pi} \int_0^\infty dy \int_{i_1'}^{i_1''} F(z) \cos y(z-t) dz \\ (202) \quad &= \frac{1}{2\pi} \int_0^\infty dy \int_{i_1'}^{i_1''} F(z) [e^{iy(z-t)} + e^{-iy(z-t)}] dz; \end{aligned}$$

that is,

$$(203) \quad f(x) = I_1 + I_2,$$

where

$$(204) \quad I_1 = \frac{1}{2\pi} \int_0^\infty e^{-iyt} \left\{ \int_{i_1'}^{i_1''} F(z) e^{iyz} dz \right\} dy,$$

and

$$(205) \quad I_2 = \frac{1}{2\pi} \int_0^\infty e^{iyt} \left\{ \int_{i_1'}^{i_1''} F(z) e^{-iyz} dz \right\} dy.$$

We shall reduce (204) to a more convenient form. The definite integral

$$\int_{i_1'}^{i_1''} F(z) e^{iyz} dz$$

is obviously a function of  $y$ .

Let  $\gamma$  be an arbitrary constant. Then

$$(206) \quad e^{iyz} = e^{-(\gamma y^2/2)} \cdot e^{iyz + (\gamma y^2/2)},$$

or

$$(207) \quad e^{iyz} = e^{-(\gamma y^2/2)} \cdot \chi(y).$$

Expanding  $\chi(y)$  into a Maclaurin's series, we have

$$(208) \quad e^{iyz} = e^{-(\gamma y^2/2)} \left\{ 1 + c_1 y i + \frac{c_2}{2} y^2 i^2 + \frac{c_3}{3} y^3 i^3 + \dots \right\},$$

where

$$c_1 i = D_y [e^{iyz + (\gamma y^2/2)}]_{y=0},$$

$$c_2 i^2 = D_y^2 [e^{iyz + (\gamma y^2/2)}]_{y=0},$$

and, in general,

$$(209) \quad c_n i^n = D_y^n [e^{iyz + (\gamma y^2/2)}]_{y=0}.$$



Therefore

$$\begin{aligned}
 c_1 &= z, \\
 c_2 &= z^2 - \gamma, \\
 &\vdots \\
 (210) \quad &\vdots \\
 &\vdots \\
 c_n &= z^n - \frac{n^{(2)}}{2} \gamma z^{n-2} + \frac{n^{(4)}}{2^2 \underline{2}} \gamma^2 z^{n-4} - \frac{n^{(6)}}{2^3 \underline{3}} \gamma^3 z^{n-6} + \dots
 \end{aligned}$$

We may now determine  $\gamma$  by imposing the condition that

$$(211) \quad \int_{i'}^v \frac{c_2}{\underline{2}} y^2 i^2 F(z) dz = 0,$$

whence

$$(212) \quad \gamma = \frac{\int_{i'}^v z^2 F(z) dz}{\int_{i'}^v F(z) dz}.$$

Introduce the notation

$$\eta_s \equiv \int_{i'}^v z^s F(z) dz,$$

and let

$$(213) \quad r\lambda_s \equiv \frac{\eta_s}{\eta_r}, \quad \text{and} \quad \lambda_s \equiv \frac{\eta_s}{\eta_0};$$

then (212) reduces to the following simple form:

$$\gamma = \lambda_2.$$

After proper substitutions we obtain the following equation:

$$\begin{aligned}
 (214) \quad I_1 &= \frac{N}{2\pi} \int_0^\infty e^{-iyt - (\lambda_2 y^2/2)} \left\{ 1 + a_1 y i + \frac{a_3}{\underline{3}} y^3 i^3 \right. \\
 &\quad \left. + \frac{a_4}{\underline{4}} y^4 i^4 + \dots \right\} dy,
 \end{aligned}$$

where

$$(215) \quad a_n = \frac{1}{N} \int_{i'}^v c_n \cdot F(z) dz$$

and

$$N = \int_{i'}^v F(z) dz = \eta_0.$$

Substitute  $-y$  for  $y$  in (205). The resulting equation is

$$(216) \quad \begin{aligned} I_2 &= -\frac{1}{2\pi} \int_0^{-\infty} e^{-iyt} \left\{ \int_{i'}^{i''} F(z) e^{iyz} dz \right\} dy \\ &= \frac{1}{2\pi} \int_{-\infty}^0 e^{-iyt} \left\{ \int_{i'}^{i''} F(z) e^{iyz} dz \right\} dy. \end{aligned}$$

It follows at once from (214) and (216) that

$$(217) \quad f(x) = \frac{N}{2\pi} \int_{-\infty}^{\infty} e^{-iyt - (\lambda_2 y^2/2)} \left\{ 1 + a_1 y i + \frac{a_3}{[3]} y^3 i^3 + \frac{a_4}{[4]} y^4 i^4 + \dots \right\}.$$

Finally, using the definite integral

$$(218) \quad \int_{-\infty}^{\infty} e^{-a^2 x^2} \cos bxdx = \frac{\sqrt{\pi}}{a} e^{-(b^2/4a^2)},$$

we have

$$(219) \quad f(x) = N \left\{ \varphi(t) - \frac{a_1}{[1]} \varphi'(t) - \frac{a_3}{[3]} \varphi^3(t) + \frac{a_4}{[4]} \varphi^4(t) - \dots \right\},$$

where

$$\begin{aligned} \varphi(t) &= \frac{1}{\sqrt{2\lambda_2\pi}} e^{-(t^2/2\lambda_2)} \\ &= \frac{1}{\sqrt{2\lambda_2\pi}} e^{-(x^2/2\lambda_2)}. \end{aligned}$$

The values of the successive coefficients are

$$(220) \quad \begin{aligned} a_1 &= \lambda_1, \\ a_3 &= \lambda_3 - 3\lambda_2\lambda_1, \\ a_4 &= \lambda_4 - 3\lambda_2^2, \\ a_5 &= \lambda_5 - 10\lambda_2\lambda_3 + 15\lambda_2^2\lambda_1, \\ a_6 &= \lambda_6 - 15\lambda_2\lambda_4 + 30\lambda_2^3. \end{aligned}$$

The function  $\lambda_s$  may be expressed in terms of the moments around the mean, for, by definition,

$$(221) \quad \lambda_s \equiv \frac{\eta_s}{\eta_0} = \frac{\int_{i'}^{i''} z^s F(z) dz}{\int_{i'}^{i''} F(z) dz} = \frac{\int_{i'}^{i''} z^s f(z^{1/q}) dz}{\int_{i'}^{i''} f(z^{1/q}) dz}.$$

Now place

$$x = z^{1/q};$$

then

$$(222) \quad \lambda_s = \frac{q \int_{l_1}^i x^{q(s+1)-1} f(x) dx}{q \int_{l_1}^i x^{q-1} f(x) dx} = \frac{\mu_{q(s+1)-1}}{\mu_{q-1}}.$$

With (222) we can now express the successive coefficients in terms of the moments around the mean thus:

$$(223) \quad \begin{aligned} a_1 &= \frac{\mu_{2q-1}}{\mu_{q-1}}, \\ a_3 &= \frac{1}{(\mu_{q-1})^2} \{ \mu_{4q-1} \mu_{q-1} - 3 \mu_{3q-1} \mu_{2q-1} \}, \\ a_4 &= \frac{1}{(\mu_{q-1})^2} \{ \mu_{5q-1} \mu_{q-1} - 3 (\mu_{3q-1})^2 \}, \\ a_5 &= \frac{1}{(\mu_{q-1})^3} \{ \mu_{6q-1} (\mu_{q-1})^2 - 10 \mu_{4q-1} \mu_{3q-1} \mu_{q-1} \\ &\quad + 15 (\mu_{3q-1})^2 \mu_{2q-1} \}, \\ a_6 &= \frac{1}{(\mu_{q-1})^3} \{ \mu_{7q-1} (\mu_{q-1})^2 - 15 \mu_{5q-1} \mu_{3q-1} \mu_{q-1} \\ &\quad + 30 (\mu_{3q-1})^3 \}. \end{aligned}$$

When  $q = 1$ , the generating function becomes

$$\frac{1}{\sqrt{2\mu_2\pi}} e^{-(x^2/2\mu_2)}$$

or

$$\frac{1}{\sigma \sqrt{2\pi}} e^{-(x^2/2\sigma^2)},$$

the well-known curve of error. And the coefficients reduce to the following forms:

$$(224) \quad \begin{aligned} a_1 &= 0, \\ a_3 &= \mu_3, \\ a_4 &= \mu_4 - 3\sigma^4, \\ a_5 &= \mu_5 - 10\sigma^2\mu_3, \\ a_6 &= \mu_6 - 15\sigma^2\mu_4 + 30\sigma^6, \end{aligned}$$

which proves that the function reduces to Charlier's Type A curve when  $q = 1$ .

The foregoing development is of very little practical value for the series-expansion obtained involves too much labor in application. Furthermore, since the coefficients of the resulting series are functions of moments of very high order even when  $q$  is small, the reliability of the coefficients might be seriously questioned. Consequently we are forced to investigate another generalization of the curve of error.

#### a. CHARLIER'S METHOD

The theoretical discussion of the method we are about to follow is found in Charlier's article entitled *Über die Darstellung willkürlicher Funktionen*. The method is too long and complicated to be reproduced here; but we have arranged the different steps in our work in the same order as in Charlier's article so that a perusal of Charlier's theoretical development will make clear to the reader the connection between the successive stages of our work.

The first step is to evaluate the  $\lambda$ -coefficients. The formula used for this purpose is<sup>1</sup>

$$(19c) \quad \lambda_s = \frac{1}{s!} \int_{-\infty}^{\infty} x^s f(x) dx.$$

Let

$$f(x) = \kappa e^{-(x^{2q}/\gamma^{2q})},$$

where  $\kappa$ ,  $q$  and  $\gamma$  are arbitrary constants.

Now determine  $\kappa$ ,  $q$  and  $\gamma$  in such a way that

$$(225) \quad \begin{aligned} \int_{-\infty}^{\infty} f(x) dx &= 1, \\ \int_{-\infty}^{\infty} x^2 f(x) dx &= \mu_2, \\ \int_{-\infty}^{\infty} x^4 f(x) dx &= \mu_4, \end{aligned}$$

where  $\mu_2$  and  $\mu_4$  are the second and the fourth moments, respectively, of the observed distribution taken about the mean.

First of all, we need to evaluate the definite integral

$$\int_0^{\infty} e^{-x^q} x^p dx,$$

<sup>1</sup> (19c) stands for equation (19) in Charlier's article.



where  $p$  is a positive integer and  $q$  is a positive real number, integral or fractional. Put

$$x^q = z,$$

then

$$qx^{q-1}dx = dz; \quad dx = \frac{1}{q}z^{(1-q)/q}dz;$$

and the integral reduces to the form

$$\frac{1}{q} \int_0^\infty e^{-xz^{[(p+1)/q-1]}} dz,$$

which is equal to

$$\frac{1}{q} \Gamma\left(\frac{p+1}{q}\right).$$

Equations (225) then reduce to the following forms:

$$(226) \quad \begin{aligned} \frac{\kappa\gamma}{q} \Gamma(1/2q) &= 1, \\ \frac{\kappa\gamma^3}{q} \Gamma(3/2q) &= \mu_2, \\ \frac{\kappa\gamma^5}{q} \Gamma(5/2q) &= \mu_4. \end{aligned}$$

Solving these equations simultaneously, we get the values of the arbitrary constants to use in the graduation of a given observed distribution. Expressing  $\gamma$  and  $\kappa$  in terms of  $\rho$  we have

$$(227) \quad \gamma = \sigma \left\{ \frac{\Gamma(\rho)}{\Gamma(3\rho)} \right\}^{1/2}; \quad \kappa = \frac{\{\Gamma(3\rho)\}^{1/2}}{2\rho\sigma\{\Gamma(\rho)\}^{3/2}};$$

$\rho$  being identical to  $1/2q$ .

Substituting successive values of  $s$  in (19c), we get

$$(228) \quad \begin{aligned} \lambda_0 &= 1, \\ \lambda_2 &= \frac{1}{2}\sigma^2, \\ \lambda_4 &= \frac{1}{[4]}\gamma^4 \frac{\Gamma(5\rho)}{\Gamma(\rho)}, \\ \lambda_6 &= \frac{1}{[6]}\gamma^6 \frac{\Gamma(7\rho)}{\Gamma(\rho)}, \\ \lambda_1 &= \lambda_3 = \lambda_5 = \dots = \lambda_{2n+1} = 0. \end{aligned}$$

Then, using Charlier's determining equations for  $S_r(x)$ , we have<sup>1</sup>

$$\begin{aligned}
 S_0(x) &= 1, \\
 S_1(x) &= -x, \\
 S_2(x) &= \frac{x^2}{2} - \frac{1}{2}\sigma^2, \\
 S_3(x) &= -\frac{x^3}{6} + \frac{1}{2}\sigma^2 \frac{x}{1}, \\
 (229) \quad S_4(x) &= \frac{x^4}{24} - \frac{1}{2}\sigma^2 \frac{x^2}{2} + \left\{ \frac{1}{4}\sigma^4 - \frac{1}{4}\gamma^4 \frac{\Gamma(5\rho)}{\Gamma(\rho)} \right\}, \\
 S_5(x) &= -\frac{x^5}{120} + \frac{1}{2}\sigma^2 \frac{x^3}{6} - \left\{ \frac{1}{4}\sigma^4 - \frac{1}{4}\gamma^4 \frac{\Gamma(5\rho)}{\Gamma(\rho)} \right\} \frac{x}{1}, \\
 S_6(x) &= \frac{x^6}{720} - \frac{1}{2}\sigma^2 \frac{x^4}{24} + \left\{ \frac{1}{4}\sigma^4 - \frac{1}{4}\gamma^4 \frac{\Gamma(5\rho)}{\Gamma(\rho)} \right\} \frac{x^2}{2} \\
 &\quad - \left\{ \frac{1}{8}\sigma^6 - \sigma^2 \frac{\gamma^4 \Gamma(5\rho)}{24 \Gamma(\rho)} + \frac{\gamma^6 \Gamma(7\rho)}{720 \Gamma(\rho)} \right\}.
 \end{aligned}$$

Finally, from (2c) we obtain the following values of the successive coefficients:

$$\begin{aligned}
 A_0 &= N, \\
 A_1 &= 0, \\
 A_2 &= 0, \\
 3 \cdot A_3 &= -N\{\mu_3\}, \\
 (230) \quad 4 \cdot A_4 &= N \left\{ \mu_4 - \gamma^4 \frac{\Gamma(5\rho)}{\Gamma(\rho)} \right\}, \\
 5 \cdot A_5 &= -N\{\mu_5 - 10\sigma^2\mu_3\}, \\
 6 \cdot A_6 &= N \left\{ \mu_6 - 15\sigma^2\mu_4 + 15\sigma^2\gamma^4 \frac{\Gamma(5\rho)}{\Gamma(\rho)} - \gamma^6 \frac{\Gamma(7\rho)}{\Gamma(\rho)} \right\},
 \end{aligned}$$

and (1c) becomes

$$(231) \quad F(x) = A_0 f(x) + A_3 f^3(x) + A_4 f^4(x) + A_5 f^5(x) + \dots,$$

where the coefficients have the values just found.

<sup>1</sup> See Appendix B for the equations determining  $S_5(x)$  and  $S_6(x)$ .

## b. CARVER'S METHOD

In a comparatively recent article, Carver<sup>1</sup> suggests the following method:

First of all he gives a mathematical development of Charlier's Type A curve, and he comes out with the following equation:

$$(232) \quad f(x) = N \left\{ \varphi(x) - \frac{c_3}{\underline{3}} \varphi^3(x) + \frac{c_4}{\underline{4}} \varphi^4(x) - \frac{c_5}{\underline{5}} \varphi^5(x) + \dots \right\}.$$

Now let  $\theta(x)$  be an arbitrarily chosen function. Expand it into a series by the use of equation (232) thus:

$$(233) \quad \theta(x) = M \left\{ \varphi(x) - \frac{d_3}{\underline{3}} \varphi^3(x) + \frac{d_4}{\underline{4}} \varphi^4(x) - \frac{d_5}{\underline{5}} \varphi^5(x) + \dots \right\},$$

where the coefficients are functions of the moments around the mean of the arbitrary function  $\theta(x)$ . By successive differentiation of both sides of (233), we may eliminate  $\varphi(x)$  between (232) and (233) and obtain the following equation:

$$(234) \quad f(x) = \frac{N}{M} \left\{ \theta(x) - \frac{a_3}{\underline{3}} \theta^3(x) + \frac{a_4}{\underline{4}} \theta^4(x) - \frac{a_5}{\underline{5}} \theta^5(x) + \dots \right\},$$

where

$$(235) \quad \begin{aligned} a_3 &= c_3 - d_3, \\ a_4 &= c_4 - d_4, \\ a_5 &= c_5 - d_5, \\ a_6 &= c_6 - d_6 - 20d_3(c_3 - d_3), \\ &\dots \end{aligned}$$

Let us apply the above method in the case where the arbitrary function is defined by the equation

$$\theta(x) = \kappa e^{-(x^2/\gamma^2)},$$

<sup>1</sup> *The Mathematical Representation of Frequency Distributions*. H. C. Carver. QUARTERLY PUBLICATION OF THE AMERICAN STATISTICAL ASSOCIATION, vol. XVII, 1921.

where  $\kappa$ ,  $q$  and  $\gamma$  are determined from the following set of simultaneous equations:

$$(236) \quad \begin{aligned} \kappa \int_{-\infty}^{\infty} e^{-(x^{2q}/\gamma^{2q})} dx &= 1, \\ \kappa \int_{-\infty}^{\infty} x^2 e^{-(x^{2q}/\gamma^{2q})} dx &= \sigma^2, \\ \kappa \int_{-\infty}^{\infty} x^4 e^{-(x^{2q}/\gamma^{2q})} dx &= \mu_4. \end{aligned}$$

By means of the formula

$$(237) \quad \int_0^{\infty} e^{-x^q} x^p dx = \frac{1}{q} \Gamma\left(\frac{p+1}{q}\right),$$

we obtain the following values of the moments around the mean of the arbitrary function:

$$(238) \quad \begin{aligned} \mu_2 &= \frac{\kappa \gamma^3}{q} \Gamma(3/2q), \\ \mu_4 &= \frac{\kappa \gamma^5}{q} \Gamma(5/2q), \\ \mu_6 &= \frac{\kappa \gamma^7}{q} \Gamma(7/2q), \\ \mu_1 = \mu_3 = \mu_5 = \dots = \mu_{2n+1} &= 0. \end{aligned}$$

But

$$\kappa = \frac{q}{\gamma \Gamma(\rho)},$$

where

$$\rho = \frac{1}{2q};$$

therefore

$$(239) \quad \begin{aligned} \mu_2 &= \gamma^2 \frac{\Gamma(3\rho)}{\Gamma(\rho)} = \sigma^2, \\ \mu_4 &= \gamma^4 \frac{\Gamma(5\rho)}{\Gamma(\rho)}, \\ \mu_6 &= \gamma^6 \frac{\Gamma(7\rho)}{\Gamma(\rho)}. \end{aligned}$$

Substituting the value of  $\gamma$  given in the following equation:

$$(240) \quad \gamma^2 = \sigma^2 \frac{\Gamma(\rho)}{\Gamma(3\rho)} = \sigma^2 [\rho]; \quad [\rho] \equiv \frac{\Gamma(\rho)}{\Gamma(3\rho)};$$



we have

$$\begin{aligned}
 \mu_2 &= \sigma^2, \\
 (241) \quad \mu_4 &= \sigma^4 [\rho]^2 \frac{\Gamma(5\rho)}{\Gamma(\rho)}, \\
 \mu_6 &= \sigma^6 [\rho]^3 \frac{\Gamma(7\rho)}{\Gamma(\rho)}.
 \end{aligned}$$

Then

$$\begin{aligned}
 d_4 &= \sigma^4 \left\{ [\rho]^2 \frac{\Gamma(5\rho)}{\Gamma(\rho)} - 3 \right\}, \\
 (242) \quad d_6 &= \sigma^6 \left\{ [\rho]^3 \frac{\Gamma(7\rho)}{\Gamma(\rho)} - 15 [\rho]^2 \frac{\Gamma(5\rho)}{\Gamma(\rho)} + 30 \right\}, \\
 d_3 &= d_5 = d_7 = \dots = d_{2n+1} = 0;
 \end{aligned}$$

accordingly

$$\begin{aligned}
 a_3 &\equiv c_3 = \mu_3, \\
 a_4 &\equiv c_4 - d_4 = \mu_4 - \sigma^4 [\rho]^2 \frac{\Gamma(5\rho)}{\Gamma(\rho)}, \\
 (243) \quad a_5 &\equiv c_5 = \mu_5 - 10\sigma^2 \mu_3, \\
 a_6 &\equiv c_6 - d_6 = \mu_6 - 15\sigma^2 \mu_4 + \sigma^6 \left\{ 15 [\rho]^2 \frac{\Gamma(5\rho)}{\Gamma(\rho)} \right. \\
 &\quad \left. - [\rho]^3 \frac{\Gamma(7\rho)}{\Gamma(\rho)} \right\}.
 \end{aligned}$$

### c. CONCLUDING REMARKS

The results obtained by both methods are absolutely identical. The new generating function is

$$f(x) = \kappa e^{-(x^2 \sigma / \gamma^2 \sigma)},$$

where

$$(244) \quad \kappa = \frac{\{\Gamma(3\rho)\}^{1/2}}{2\rho\sigma\{\Gamma(\rho)\}^{3/2}}; \quad \gamma = \sigma \left\{ \frac{\Gamma(\rho)}{\Gamma(3\rho)} \right\}^{1/2}.$$

The resulting series-expansion is

$$(245) \quad F(x) = N \left\{ f(x) - \frac{a_3}{[3]} f^3(x) + \frac{a_4}{[4]} f^4(x) - \dots \right\},$$

where

$$\begin{aligned}
 a_3 &= \mu_3, \\
 a_4 &= \mu_4 - \sigma^4 [\rho]^2 \frac{\Gamma(5\rho)}{\Gamma(\rho)}, \\
 a_5 &= \mu_5 - 10\sigma^2 \mu_3, \\
 a_6 &= \mu_6 - 15\sigma^2 \mu_4 + \sigma^6 \left\{ 15[\rho]^2 \frac{\Gamma(5\rho)}{\Gamma(\rho)} - [\rho]^3 \frac{\Gamma(7\rho)}{\Gamma(\rho)} \right\}.
 \end{aligned}
 \tag{246}$$

If we place  $q$  equal to 1, the generating function becomes the normal curve of error, and our constants and coefficients therefore should reduce to the same constants and coefficients that are connected with Charlier's Type A curve. Such is the case as will now be shown.

When  $q = 1$ ,

$$\rho \equiv \frac{1}{2q} = 1/2,
 \tag{247}$$

then

$$\begin{aligned}
 \kappa &= \frac{\{\Gamma(\frac{3}{2})\}^{1/2}}{2 \cdot \frac{1}{2} \cdot \sigma \{\Gamma(\frac{1}{2})\}^{3/2}} = \frac{\sqrt{\frac{1}{2}} \{\Gamma(\frac{1}{2})\}^{1/2}}{\sigma \{\Gamma(\frac{1}{2})\}^{3/2}} = \frac{1}{\sigma \sqrt{2\pi}}; \\
 \gamma &= \sigma \left\{ \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})} \right\}^{1/2} = \sigma \left\{ \frac{1}{\frac{1}{2}} \right\}^{1/2} = \sigma \cdot \sqrt{2}.
 \end{aligned}
 \tag{248}$$

These are the constants of the normal curve of error.

The coefficients reduce to the following forms:

$$\begin{aligned}
 a_3 &= \mu_3, \\
 a_4 &= \mu_4 - \sigma^4 \left\{ 4 \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{1}{2})} \right\} = \mu_4 - \sigma^4 \{ 4 \cdot \frac{3}{2} \cdot \frac{1}{2} \} \\
 &= \mu_4 - 3\sigma^4, \\
 a_5 &= \mu_5 - 10\sigma^2 \mu_3, \\
 a_6 &= \mu_6 - 15\sigma^2 \mu_4 + \sigma^6 \{ 15 \cdot 4 \cdot \frac{3}{2} \cdot \frac{1}{2} - 8 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \} \\
 &= \mu_6 - 15\sigma^2 \mu_4 + 30\sigma^6.
 \end{aligned}
 \tag{249}$$

Thus we see that the coefficients also would be identical to those of Charlier's Type A curve when we make  $q$  equal to 1. We seem justified therefore in stating that the general curve we have found represents a family of curves of which Charlier's Type A curve is a member.

Pearson's  $\beta_2$ -function may now be calculated to determine which value of  $q$  to use in graduating a given observed frequency distribution. That is, we have to evaluate  $\alpha_4$ , since, by hypothesis, the two are identical. We have, by definition,

$$\alpha_4 \equiv \frac{\mu_4}{\mu_2^2},$$

accordingly, substituting the values of  $\mu_2$  and  $\mu_4$ , respectively, we have

$$(250) \quad \alpha_4 = \frac{\Gamma(5\rho)\Gamma(\rho)}{\{\Gamma(3\rho)\}^2}.$$

A reference table is given in Appendix C, where the corresponding values of

$$\rho, \log \Gamma(\rho), \log \Gamma(3\rho), \log \Gamma(5\rho), \text{ and } \log \Gamma(7\rho)$$

are given for different values of  $\alpha_4$ .

It must be noted that the value of  $\alpha_4$  is 3 when  $q$  is equal to 1. The quantity  $q$  may be expressed as a quotient of two positive integers  $a$  and  $b$ , if  $b$  is assumed to be odd. This last limitation is necessary to avoid imaginary quantities in the resulting graduation. That is,

$$q = a/b$$

and, from the assumption regarding  $a$  and  $b$ , it is obvious that  $q$  has an infinite degree of freedom, and hence the curve we have deduced is fairly general.

### *Third Numerical Example*

The following table gives the head-breadths of 1000 Cambridge men. It is taken from *Biometrika*, volume I, 1901-02, page 220.

| x-inches | Frequency | x-inches | Frequency |
|----------|-----------|----------|-----------|
| 5.5..... | 3         | 6.2..... | 142       |
| 5.6..... | 12        | 6.3..... | 99        |
| 5.7..... | 43        | 6.4..... | 37        |
| 5.8..... | 80        | 6.5..... | 15        |
| 5.9..... | 131       | 6.6..... | 12        |
| 6.0..... | 236       | 6.7..... | 3         |
| 6.1..... | 185       | 6.8..... | 2         |

The  $\alpha$ 's of this distribution were found to be

$$(251) \quad \begin{aligned} \alpha_2 &= 1.0, \\ \alpha_3 &= .26783608, \\ \alpha_4 &= 3.41557774, \\ \alpha_5 &= 3.1318036. \end{aligned}$$

Referring to the reference table for the generalized curve of error (Table I, Appendix C), we find that the value of  $\alpha_4$  nearest to the observed  $\alpha_4$  corresponds to the case when  $\rho = .59$ .

Therefore, for our generating function, we use

$$(252) \quad \theta(x) = \kappa e^{-(x/\gamma)^{100/59}},$$

where  $\kappa$  and  $\gamma$  are arbitrary constants defined by (244).

We are now ready for the graduation proper. The following expression for the third derivative of the generating functions is used in the evaluation of the second term of the series:

$$(253) \quad \frac{\theta^3(x)}{\theta(x)} = \frac{1}{\gamma^3} \left[ -u(u-1)(u-2) \left(\frac{x}{\gamma}\right)^{u-3} + 3u^2(u-1) \left(\frac{x}{\gamma}\right)^{2u-3} - u^3 \left(\frac{x}{\gamma}\right)^{3u-3} \right],$$

where

$$u = \frac{100}{59}.$$

The following table gives the final results of the graduation.

TABLE 5

DISTRIBUTION OF HEAD-BREADTHS OF 1000 CAMBRIDGE MEN  
*Third Numerical Example*

*Source of Rough Data:* BIOMETRIKA, vol. I, p. 220.

*Generating Function Employed:*  $\varphi(x) = \kappa e^{-(x/\gamma)^{100/69}}$ .

| Mid-Point of<br>Class Interval | Observed<br>Frequency | Graduation    |                |
|--------------------------------|-----------------------|---------------|----------------|
|                                |                       | One Term Used | Two Terms Used |
|                                |                       | First         | Second         |
| (1)                            | (2)                   | (3)           | (4)            |
| Inches                         |                       |               |                |
| 5.4                            | 0                     | 2             | 1              |
| 5.5                            | 3                     | 5             | 3              |
| 5.6                            | 12                    | 14            | 12             |
| 5.7                            | 43                    | 36            | 34             |
| 5.8                            | 80                    | 77            | 81             |
| 5.9                            | 131                   | 139           | 152            |
| 6.0                            | 236                   | 201           | 227            |
| 6.1                            | 185                   | 211           | 182            |
| 6.2                            | 142                   | 154           | 142            |
| 6.3                            | 99                    | 90            | 87             |
| 6.4                            | 37                    | 44            | 46             |
| 6.5                            | 15                    | 18            | 21             |
| 6.6                            | 12                    | 6             | 8              |
| 6.7                            | 3                     | 2             | 3              |
| 6.8                            | 2                     | 1             | 1              |

The foregoing table is inconsistent for we are comparing areas (the original data) and ordinates (the graduated data). Table 5a is given as an improvement. The graduated data here are obtained from Table 5 by the use of the quadrature formula:

$$(254) \quad \int_{-1/2}^{1/2} y dx = \frac{1}{24} [y_{-1} + 22y_0 + y_{+1}].$$



TABLE 5a

DISTRIBUTION OF HEAD-BREADTHS OF 1000 CAMBRIDGE MEN

*Third Numerical Example**Source of Rough Data:* BIOMETRIKA, vol. I, p. 220.*Generating Function Employed:*  $\varphi(x) = ke^{-(x/\gamma)^{100/59}}$ .

| Mid-Point of<br>Class Interval | Observed<br>Frequency | Graduation    |                |
|--------------------------------|-----------------------|---------------|----------------|
|                                |                       | One Term Used | Two Terms Used |
|                                |                       | First         | Second         |
| (1)                            | (2)                   | (3)           | (4)            |
| Inches                         |                       |               |                |
| 5.4                            | 0                     | 2             | 1              |
| 5.5                            | 3                     | 5             | 3              |
| 5.6                            | 12                    | 15            | 12             |
| 5.7                            | 43                    | 37            | 35             |
| 5.8                            | 80                    | 78            | 81             |
| 5.9                            | 131                   | 139           | 152            |
| 6.0                            | 236                   | 199           | 222            |
| 6.1                            | 185                   | 208           | 182            |
| 6.2                            | 142                   | 154           | 142            |
| 6.3                            | 99                    | 91            | 88             |
| 6.4                            | 37                    | 44            | 47             |
| 6.5                            | 15                    | 18            | 22             |
| 6.6                            | 12                    | 7             | 9              |
| 6.7                            | 3                     | 2             | 3              |
| 6.8                            | 2                     | 1             | 1              |

### CHAPTER THREE

The numerical example just given shows one difficulty in employing generating functions which are not integrable. In order to make a fair comparison between the original data and the results of the graduation, we have to reduce graduated ordinates to areas by means of a quadrature formula which is, at best, merely an approximation and is therefore subject to errors. If the generating function used was integrable, the final step would have been unnecessary. Graduated areas could then be obtained directly from the graduating function, and there would not be any need of resorting to some sort of approximation.

One such function is  $\text{sech } x$ . Another is  $\text{sech}^2 x$ . Besides being integrable, both these functions are unimodal; and they, together with their successive derivatives, vanish at  $x = \pm \infty$ . Each rises to a maximum point at  $x = 0$ , and then slopes down on both sides of the  $y$ -axis and eventually becomes asymptotic to the axis of reals.

Starting again with Fourier's double integral, we have

$$\begin{aligned}
 f(x) &= \frac{1}{\pi} \int_0^\infty dy \int_{-l_1}^l f(z) \cos y(z-x) dz \\
 (301) \quad &= \frac{1}{\pi} \int_0^\infty dy \int_{-l_1}^l f(z) \cdot \frac{1}{2} \{e^{iy(z-x)} + e^{-iy(z-x)}\} dz \\
 &= \frac{1}{2\pi} \int_{-\infty}^\infty e^{-iyx} \left\{ \int_{-l_1}^l f(z) e^{iyz} dz \right\} dy.
 \end{aligned}$$

Now let  $\gamma_1$  and  $\gamma_2$  be two arbitrary constants which, for the time being, are undetermined. We may write

$$e^{iyz} = \frac{e^{\gamma_1 y i}}{e^{\gamma_2 y} + e^{-\gamma_2 y}} \cdot \chi(y),$$

where

$$(302) \quad \chi(y) \equiv e^{iy(z-\gamma_1)} \{e^{\gamma_2 y} + e^{-\gamma_2 y}\}.$$

Expand  $\chi(y)$  by Maclaurin's theorem, and obtain the following

power series in  $y$ :

$$\begin{aligned}
 \chi(y) = & 2 + 2i(z - \gamma_1)y \\
 & + [ \{i(z - \gamma_1) + \gamma_2\}^2 + \{i(z - \gamma_1) - \gamma_2\}^2 ] \frac{y^2}{2} \\
 & + \cdots + [ \{i(z - \gamma_1) + \gamma_2\}^n \\
 & + \{i(z - \gamma_1) - \gamma_2\}^n ] \frac{y^n}{n} + \cdots.
 \end{aligned}
 \tag{303}$$

The region of convergence of this series is the whole finite plane. Whence

$$\begin{aligned}
 \int_{-l_1}^l f(z) e^{iyz} dz &= \frac{e^{\gamma_1 y i}}{e^{\gamma_2 y} + e^{-\gamma_2 y}} \int_{-l_1}^l f(z) \chi(y) dz \\
 &= \frac{2e^{\gamma_1 y i}}{e^{\gamma_2 y} + e^{-\gamma_2 y}} \left\{ N + c_1 y i + \frac{c_2}{2} y^2 i^2 \right. \\
 &\quad \left. + \cdots + \frac{c_n}{n} y^n i^n + \cdots \right\},
 \end{aligned}
 \tag{304}$$

where

$$N = \int_{-l_1}^l f(z) dz$$

and

$$\begin{aligned}
 c_n = \frac{1}{2} \int_{-l_1}^l f(z) [ \{i(z - \gamma_1) + \gamma_2\}^n \\
 + \{i(z - \gamma_1) - \gamma_2\}^n ] dz.
 \end{aligned}
 \tag{305}$$

Now determine the arbitrary constants  $\gamma_1$  and  $\gamma_2$  in such a manner as to make

$$c_1 = c_2 = 0,$$

that is

$$\int_{-l_1}^l (z - \gamma_1) f(z) dz = \int_{-l_1}^l [ - (z - \gamma_1)^2 + \gamma_2^2 ] f(z) dz = 0.
 \tag{306}$$

Solving equations (306), we obtain the following values of the two arbitrary constants:

$$\begin{aligned}
 \gamma_1 &= \frac{\int_{-l_1}^l z f(z) dz}{\int_{-l_1}^l f(z) dz}, \\
 \gamma_2^2 &= \frac{\int_{-l_1}^l (z - \gamma_1)^2 f(z) dz}{\int_{-l_1}^l f(z) dz}.
 \end{aligned}
 \tag{307}$$

Or, using statistical terms, we may say that

$\gamma_1$  = the distance of the mean from the arbitrary origin,

$\gamma_2$  = the standard deviation.

Designate these by the usual notation

$$(308) \quad \gamma_1 = b, \quad \gamma_2 = \sigma,$$

and substitute in (301); it now becomes

$$(309) \quad f(x) = \frac{N}{2\pi} \int_{-\infty}^{\infty} \frac{2e^{-iy(x-b)}}{e^{\sigma y} + e^{-\sigma y}} \left\{ 1 + \frac{a_3}{|\underline{3}|} y^3 i^3 + \frac{a_4}{|\underline{4}|} y^4 i^4 + \dots \right\} dy,$$

where

$$(310) \quad a_n = \frac{c_n}{N}.$$

But <sup>1</sup>

$$(311) \quad \begin{aligned} \int_{-\infty}^{\infty} \frac{2e^{-iy(x-b)}}{e^{\sigma y} + e^{-\sigma y}} dy &= \int_{-\infty}^0 \frac{2e^{-iy(x-b)}}{e^{\sigma y} + e^{-\sigma y}} dy + \int_0^{\infty} \frac{2e^{-iy(x-b)}}{e^{\sigma y} + e^{-\sigma y}} dy \\ &= 2 \int_0^{\infty} \frac{e^{iy(x-b)} + e^{-iy(x-b)}}{e^{\sigma y} + e^{-\sigma y}} dy \\ &= 4 \int_0^{\infty} \frac{\cos y(x-b)}{e^{\sigma y} + e^{-\sigma y}} dy \\ &= 4 \cdot \frac{\pi}{2\sigma} \frac{1}{e^{(x-b)\pi/2\sigma} + e^{-[(x-b)\pi]/2\sigma}} \\ &= \frac{\pi}{\sigma} \operatorname{sech} \frac{(x-b)\pi}{2\sigma}. \end{aligned}$$

It follows from (311) that equation (309) may be reduced to the following form:

$$(312) \quad f(x) = N \left\{ F(x) - \frac{a_3}{|\underline{3}|} F^3(x) + \frac{a_4}{|\underline{4}|} F^4(x) - \frac{a_5}{|\underline{5}|} F^5(x) + \dots \right\},$$

where

$$F(x) = \frac{1}{2\sigma} \operatorname{sech} \frac{(x-b)\pi}{2\sigma}.$$

<sup>1</sup> *Nouvelles Tables d'Intégrales Définies*. Bicrens de Haan. 1867. Table 264, No. 14.

When the origin is taken at the mean, the generating function takes the following form:

$$F(x) = \frac{1}{2\sigma} \operatorname{sech} \frac{x\pi}{2\sigma}.$$

We will next determine the values of the coefficients  $a_3, a_4, a_5, \dots, a_n$ . We have

$$\begin{aligned} a_n i^n &= \frac{1}{2N} \int_{-l_1}^l f(z) \{ (\overline{iz - \gamma_1} + \gamma_2)^n \\ &\quad + (\overline{iz - \gamma_1} - \gamma_2)^n \} dz \\ (313) \quad &= \frac{i^n}{2N} \int_{-l_1}^l f(z) \left\{ (z - \gamma_1)^n - \frac{n^{(2)}}{2} \gamma_2^2 (z - \gamma_1)^{n-2} \right. \\ &\quad \left. + \frac{n^{(4)}}{4} \gamma_2^4 (z - \gamma_1)^{n-4} + \dots \right\} dz. \end{aligned}$$

Accordingly

$$(314) \quad a_n = \mu_n - \frac{n^{(2)}}{2} \sigma^2 \mu_{n-2} + \frac{n^{(4)}}{4} \sigma^4 \mu_{n-4} - \dots$$

Giving  $n$  the values 3, 4, 5,  $\dots$  successively, we obtain from (314) the following values of the coefficients:

$$\begin{aligned} a_3 &= \mu_3, \\ a_4 &= \mu_4 - 5\sigma^4, \\ a_5 &= \mu_5 - 10\sigma^2\mu_3, \\ a_6 &= \mu_6 - 15\sigma^2\mu_4 + 14\sigma^6, \\ (315) \quad a_7 &= \mu_7 - 21\sigma^2\mu_5 + 35\sigma^4\mu_3, \\ a_8 &= \mu_8 - 28\sigma^2\mu_6 + 70\sigma^4\mu_4 - 27\sigma^8, \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

It would be enlightening to compare these coefficients with those obtained by Charlier's method and those found by Carver's process of elimination. We shall compare the coefficients up to and including  $a_6$  only for the amount of work involved in calculating them by either method is considerable. Charlier, by an ingenious device, has calculated the coefficient up to the fourth. We shall here compute the first six coefficients



using the general method suggested by him and thus check the values he obtained by the use of a special device.

#### a. CHARLIER'S METHOD

As stated in Chapter Two, the first step in Charlier's method is evaluating what he calls the  $\lambda$ -coefficients defined by the following equation:

$$\begin{aligned} \lambda_s &= \int_{-\infty}^{\infty} x^s \varphi(x) dx \\ (316) \quad &= \frac{2}{\pi \sigma_1} \int_{-\infty}^{\infty} \frac{x^s dx}{e^{x/\sigma_1} + e^{-(x/\sigma_1)}}. \end{aligned}$$

It must be noted, before we go any further, that Charlier uses for his generating function

$$\varphi(x) = \frac{1}{\pi \sigma_1} \operatorname{sech} \frac{x}{\sigma_1},$$

while ours is

$$F(x) = \frac{1}{2\sigma} \operatorname{sech} \frac{\pi}{2\sigma} x.$$

If in  $\varphi(x)$  we substitute  $\sigma$  for  $\pi \sigma_1/2$  we obtain our generating function.

Introduce for convenience the notation

$$(317) \quad \int_{(s)} \equiv \int_{-\infty}^{\infty} \frac{x^s dx}{e^{x/\sigma_1} + e^{-(x/\sigma_1)}};$$

then, from the definite integral<sup>1</sup>

$$(318) \quad \int_{-\infty}^{\infty} \frac{x^{2a} dx}{e^{px} + e^{-px}} = \left( \frac{\pi}{2p} \right)^{2a+1} B_{2a}$$

we obtain the following values:

$$\begin{aligned} \int_{(0)} &= \frac{\pi \sigma_1}{2}, \\ \int_{(2)} &= \frac{1}{2^3} \pi^3 \sigma_1^3, \\ (319) \quad \int_{(4)} &= \frac{5}{2^5} \pi^5 \sigma_1^5, \\ \int_{(6)} &= \frac{61}{2^7} \pi^7 \sigma_1^7, \end{aligned}$$

<sup>1</sup> *Ibid.* Table 84, No. 12.

whence

$$\begin{aligned}
 \lambda_0 &= \frac{2}{\pi\sigma_1} \cdot \frac{\pi\sigma_1}{2} = 1, \\
 \lambda_2 &= \frac{1}{\pi\sigma_1} \cdot \frac{1}{2^3} \pi^3 \sigma_1^3 = \frac{1}{8} \pi^2 \sigma_1^2, \\
 \lambda_4 &= \frac{5}{4 \cdot 2^4} \pi^4 \sigma_1^4 = \frac{5}{384} \pi^4 \sigma_1^4, \\
 \lambda_6 &= \frac{61}{6 \cdot 2^6} \pi^6 \sigma_1^6 = \frac{61}{46080} \pi^6 \sigma_1^6, \\
 &\vdots \\
 \lambda_1 &= \lambda_3 = \lambda_5 = \dots = \lambda_{2n+1} = 0.
 \end{aligned}
 \tag{320}$$

If we substitute in (320)  $\sigma$  for  $\pi\sigma_1/2$ , we get the following values of the  $\lambda$ -coefficients corresponding to our generating function:

$$\begin{aligned}
 \lambda_0 &= 1; & \lambda_2 &= \frac{1}{2} \sigma^2; \\
 \lambda_4 &= \frac{5}{24} \sigma^4; & \lambda_6 &= \frac{61}{720} \sigma^6; \\
 \lambda_1 &= \lambda_3 = \lambda_5 = \dots = \lambda_{2n+1} = 0.
 \end{aligned}
 \tag{321}$$

Having found the successive values of  $\lambda_s$ , it is a simple matter to obtain the values of  $S_r(x)$  up to  $r = 6$ . They are

$$\begin{aligned}
 S_0(x) &= 1, \\
 S_1(x) &= -x, \\
 S_2(x) &= \frac{x^2}{2} - \frac{1}{2} \sigma^2, \\
 S_3(x) &= -\frac{x^3}{3} + \frac{1}{2} \sigma^2 \frac{x}{1}, \\
 S_4(x) &= \frac{x^4}{4} - \frac{1}{2} \sigma^2 \frac{x^2}{2} + \frac{1}{24} \sigma^4, \\
 S_5(x) &= -\frac{x^5}{5} + \frac{1}{2} \sigma^2 \frac{x^3}{3} - \frac{1}{24} \sigma^4 \frac{x}{1}, \\
 S_6(x) &= \frac{x^6}{6} - \frac{1}{2} \sigma^2 \frac{x^4}{4} + \frac{1}{24} \sigma^4 \frac{x^2}{2} - \frac{1}{720} \sigma^6.
 \end{aligned}
 \tag{322}$$

Substituting the different values of  $S_s(x)$  in the equation

$$(323) \quad A_s = \int_{-\infty}^{\infty} f(x) \cdot S_s(x) dx,$$

we have the following values of  $A_s$ , remembering that

$$\mu_s = \frac{\int_{-\infty}^{\infty} x^s f(x) dx}{\int_{-\infty}^{\infty} f(x) dx},$$

$$\mu_1 = 0, \quad \text{and} \quad \mu_2 = \sigma^2,$$

$$A_0 = 1 = a_0,$$

$$(324) \quad \underline{3} \cdot A_3 = \mu_3 = a_3,$$

$$\underline{4} \cdot A_4 = \mu_4 - 5\sigma^4 = a_4,$$

$$\underline{5} \cdot A_5 = \mu_5 - 10\sigma^2\mu_3 = a_5,$$

$$\underline{6} \cdot A_6 = \mu_6 - 15\sigma^2\mu_4 + 14\sigma^6 = a_6.$$

#### b. CARVER'S METHOD

Let the arbitrary function  $\theta(x)$  be defined by the equation

$$(325) \quad \theta(x) = \frac{\kappa}{e^{c_1 x} + e^{-c_1 x}} = \frac{\kappa}{2} \operatorname{sech} c_1 x,$$

where  $\kappa$  and  $c_1$  are the solutions of the simultaneous equations

$$(326) \quad \int_{-\infty}^{\infty} \theta(x) dx = 1,$$

$$\int_{-\infty}^{\infty} x^2 \theta(x) dx = \sigma^2.$$

But <sup>1</sup>

$$(327) \quad \int_0^{\infty} \frac{dx}{e^{c_1 x} + e^{-c_1 x}} = \frac{\pi}{4c_1};$$

therefore

$$(328) \quad \int_{-\infty}^{\infty} \theta(x) dx = \frac{\kappa\pi}{4c_1} = 1;$$

accordingly

$$\kappa = \frac{2c_1}{\pi}.$$

<sup>1</sup> A *Short Table of Integrals*. B. O. Peirce. Formula No. 498.

Also<sup>1</sup>

$$(329) \quad \int_0^{\infty} \frac{x^2 dx}{e^x + e^{-x}} = \frac{1}{16} \pi^3,$$

from which we get

$$(330) \quad \int_{-\infty}^{\infty} x^2 \theta(x) dx = \frac{\kappa}{c_1^3 8} \pi^3 = \sigma^2.$$

Solving, we have

$$(331) \quad c_1 = \frac{\pi}{2\sigma}; \quad \kappa = \frac{1}{\sigma}.$$

Therefore, from (325) and (331)

$$(332) \quad \theta(x) = \frac{1}{\sigma} \frac{1}{e^{\pi x/2\sigma} + e^{-(\pi x/2\sigma)}}.$$

Using again the formula

$$(333) \quad \int_0^{\infty} \frac{x^{2a} dx}{e^{px} + e^{-px}} = \frac{1}{2} \left( \frac{\pi}{2p} \right)^{2a+1} B_{2a},$$

we obtain the following values of the moments around the mean:

$$(334) \quad \begin{aligned} \mu_2 &= \sigma^2; & \mu_4 &= 5\sigma^4; & \mu_6 &= 61\sigma^6; \\ \mu_1 &= \mu_3 = \mu_5 = \dots = \mu_{2n+1} &= 0. \end{aligned}$$

whence

$$(335) \quad \begin{aligned} d_4 &= 2\sigma^4; & d_6 &= 16\sigma^6; \\ d_3 &= d_5 = \dots = d_{2n+1} &= 0. \end{aligned}$$

From these we get the following values of the successive coefficients:

$$(336) \quad \begin{aligned} a_3 &= c_3 &= \mu_3, \\ a_4 &= c_4 - d_4 &= \mu_4 - 5\sigma^4, \\ a_5 &= c_5 &= \mu_5 - 10\sigma^2\mu_3, \\ a_6 &= c_6 - d_6 &= \mu_6 - 15\sigma^2\mu_4 + 14\sigma^6. \end{aligned}$$

These are absolutely identical to those found by Charlier's method and the ones we have obtained earlier in this chapter by the use of a Fourier's double integral.

A general method for finding the  $n$ th derivative of  $\text{sech}(x-b)\pi/2\sigma$  may be found useful. By differentiation,

<sup>1</sup> Bierens de Haan. Loc. cit. Table 84, No. 12.

$$\begin{aligned}
 f(x) &= \operatorname{sech} \frac{(x-b)\pi}{2\sigma}, \\
 f'(x) &= -\frac{\pi}{2\sigma} \left\{ \tanh \frac{(x-b)\pi}{2\sigma} \right\} f(x), \\
 f^2(x) &= \left( \frac{\pi}{2\sigma} \right)^2 \left\{ 2 \tanh^2 \frac{(x-b)\pi}{2\sigma} - 1 \right\} f(x), \\
 (337) \quad f^3(x) &= -\left( \frac{\pi}{2\sigma} \right)^3 \left\{ 6 \tanh^3 \frac{(x-b)\pi}{2\sigma} \right. \\
 &\quad \left. - 5 \tanh \frac{(x-b)\pi}{2\sigma} \right\} f(x), \\
 f^4(x) &= \left( \frac{\pi}{2\sigma} \right)^4 \left\{ 24 \tanh^4 \frac{(x-b)\pi}{2\sigma} \right. \\
 &\quad \left. - 28 \tanh^2 \frac{(x-b)\pi}{2\sigma} + 5 \right\} f(x).
 \end{aligned}$$

and, in general,

$$\begin{aligned}
 (338) \quad f^{2n}(x) &= \kappa^{2n} \left\{ \sum_{j=0}^{j=n} A_{2n, 2j} \tanh^{2j} \kappa(x-b) \right\} f(x), \\
 f^{2n+1}(x) &= \kappa^{2n+1} \left\{ \sum_{j=0}^{j=n} A_{2n+1, 2j+1} \tanh^{2j+1} \kappa(x-b) \right\} f(x),
 \end{aligned}$$

where  $\kappa = \pi/2\sigma$ .

Table 6 gives the values of  $A_{r, s}$  for  $r = 1, 2, 3, 4, 5$ .

TABLE 6

TABLE TO FACILITATE THE DETERMINATION OF THE  $r$ TH DERIVATIVE OF  $\operatorname{SECH} t$

|     | (-1) | (0) | (1)        | (2) | (3) | (4)        | (5)  | (6) | (7) |
|-----|------|-----|------------|-----|-----|------------|------|-----|-----|
| (0) | 0    | 1   | 0          | 0   | 0   | 0          | 0    | 0   | 0   |
| (1) | 0    | 0   | -1         | 0   | 0   | 0          | 0    | 0   | 0   |
| (2) | 0    | -1  | 0          | 2   | 0   | 0          | 0    | 0   | 0   |
| (3) | 0    | 0   | 5          | 0   | -6  | 0          | 0    | 0   | 0   |
| (4) | 0    | 5   | 0          | -28 | 0   | 24         | 0    | 0   | 0   |
| (5) | 0    | 0   | -61        | 0   | 180 | 0          | -120 | 0   | 0   |
| (6) |      |     |            |     |     | $A_{6, 4}$ |      |     |     |
| (7) |      |     | $A_{7, 1}$ |     |     |            |      |     |     |



To find  $A_{r,s}$ , the numerical coefficient of  $\tanh^s \kappa(x-b)$  in the  $r$ th derivative of  $\operatorname{sech} \kappa(x-b)$ , we look for the number determined by line  $(r)$  and column  $(s)$  of the above table. For example, for the numerical coefficient of  $\tanh^2 \kappa(x-b)$  in the fourth derivative of  $\operatorname{sech} \kappa(x-b)$  we take the number on line (4) and under column (2), which is  $-28$ . It is evident that with the above table we can compute at once the first five derivatives of  $\operatorname{sech} \kappa(x-b)$ .

Table 6 may be easily extended by the use of the following recurring formula:

$$(339) \quad A_{r+1,s} = \{(s+1)A_{r,s+1} - sA_{r,s-1}\},$$

with the restrictions that

$$\begin{aligned} A_{m,-1} &= 0, \\ A_{m,s} &= 0; \quad s > m. \end{aligned}$$

This recurring relation is obtained by mathematical induction, and it holds for even or odd values of  $r$ .

If we slightly modify the first part of the preceding discussion, we would arrive at a different generating function. Let

$$\begin{aligned} (340) \quad e^{izy} &= \frac{e^{\gamma_1 y i}}{e^{\gamma_2 y} - e^{-\gamma_2 y}} \cdot e^{iy(z-\gamma_1)} \{e^{\gamma_2 y} - e^{-\gamma_2 y}\} \\ &= \frac{e^{\gamma_1 y i}}{e^{\gamma_2 y} - e^{-\gamma_2 y}} \cdot \chi(y), \end{aligned}$$

where now

$$(341) \quad \chi(y) = e^{iy(z-\gamma_1)} \{e^{\gamma_2 y} - e^{-\gamma_2 y}\}.$$

This may be expanded into the following Maclaurin's series:

$$\begin{aligned} (342) \quad \chi(y) &= 2\gamma_2 y + \{(\overline{iz - \gamma_1 + \gamma_2})^2 \\ &\quad - (\overline{iz - \gamma_1 - \gamma_2})^2\} \frac{y^2}{[2]} + \dots \\ &\quad + \{(\overline{iz - \gamma_1 + \gamma_2})^n \\ &\quad - (\overline{iz - \gamma_1 - \gamma_2})^n\} \frac{y^n}{[n]} + \dots \end{aligned}$$

and it follows that

$$\begin{aligned} (343) \quad \int_{-l_1}^l f(z) e^{izy} dz &= \int_{-l_1}^l f(z) \frac{e^{\gamma_1 y i}}{e^{\gamma_2 y} - e^{-\gamma_2 y}} \cdot \chi(y) dz \\ &= \frac{e^{\gamma_1 y i}}{e^{\gamma_2 y} - e^{-\gamma_2 y}} \int_{-l_1}^l f(z) \cdot \chi(y) dz. \end{aligned}$$

Or

$$(344) \quad \int_{-l_1}^l f(z) e^{iuz} dz = \frac{2ye^{\gamma_1 u i}}{e^{\gamma_2 u} - e^{-\gamma_2 u}} \left\{ N + \frac{c_2}{[2]} yi + \frac{c_3}{[3]} y^2 i^2 + \dots \right\},$$

where

$$N = \int_{-l_1}^l f(z) dz$$

and

$$(345) \quad c_n i^{n-1} = \frac{1}{2} \int_{-l_1}^l f(z) \cdot \{ (iz - \gamma_1 + \gamma_2)^n - (\overline{iz - \gamma_1 - \gamma_2})^n \} dz.$$

Imposing, as before, the conditions that

$$c_2 = c_3 = 0,$$

we obtain the following equations with which to evaluate the two arbitrary constants  $\gamma_1$  and  $\gamma_2$ :

$$(346) \quad \int_{-l_1}^l (z - \gamma_1) f(z) dz = 0,$$

$$\int_{-l_1}^l \{ \gamma_2^2 - 3(z - \gamma_1)^2 \} f(z) dz = 0.$$

Solving, we have

$$(347) \quad \gamma_1 = \frac{\int_{-l_1}^l z f(z) dz}{\int_{-l_1}^l f(z) dz} = b,$$

$$\gamma_2^2 = \frac{3 \int_{-l_1}^l (z - b)^2 f(z) dz}{\int_{-l_1}^l f(z) dz};$$

$$\gamma_2 = \sqrt{3}\sigma.$$

Whence, substituting in (301),

$$(348) \quad f(x) = \frac{N}{2\pi} \int_{-\infty}^{\infty} \frac{2\gamma_2 y e^{-iy(x-b)}}{e^{\gamma_2 y} - e^{-\gamma_2 y}} \left\{ 1 + \frac{a_4}{[4]} y^3 i^3 + \frac{a_5}{[5]} y^4 i^4 + \dots \right\} dy,$$

where

$$(349) \quad a_n = n\mu_{n-1} - \frac{n^{(3)}}{[3]} \gamma_2^2 \mu_{n-3} + \frac{n^{(5)}}{[5]} \gamma_2^4 \mu_{n-5} - \dots$$

But

$$(350) \quad \int_{-\infty}^{\infty} \frac{2\gamma_2 y e^{-iy(x-b)}}{e^{\gamma_2 y} - e^{-\gamma_2 y}} dy = \frac{2}{\gamma_2} \int_{-\infty}^{\infty} \frac{y e^{-(iy/\gamma_2)(x-b)}}{e^y - e^{-y}} dy;$$

$$(351) \quad \begin{aligned} & \int_{-\infty}^{\infty} \frac{y e^{-(iy/\gamma_2)(x-b)}}{e^y - e^{-y}} dy \\ &= \int_{-\infty}^0 \frac{y e^{-(iy/\gamma_2)(x-b)}}{e^y - e^{-y}} dy + \int_0^{\infty} \frac{y e^{-(iy/\gamma_2)(x-b)}}{e^y - e^{-y}} dy \\ &= \int_0^{\infty} y \frac{\{e^{(iy/\gamma_2)(x-b)} + e^{-(iy/\gamma_2)(x-b)}\}}{e^y - e^{-y}} dy \\ &= 2 \int_0^{\infty} y \frac{\cos \frac{y}{\gamma_2} (x-b)}{e^y - e^{-y}} dy; \end{aligned}$$

and <sup>1</sup>

$$(352) \quad \int_0^{\infty} y \frac{\cos \frac{y}{\gamma_2} (x-b)}{e^y - e^{-y}} dy = \frac{\pi^2}{2} \frac{e^{-(\pi/\gamma_2)(x-b)}}{\{1 + e^{-(\pi/\gamma_2)(x-b)}\}^2}.$$

Therefore

$$(353) \quad \begin{aligned} & \frac{2}{\gamma_2} \int_{-\infty}^{\infty} \frac{y e^{-(iy/\gamma_2)(x-b)}}{e^y - e^{-y}} dy \\ &= \frac{2\pi^2}{\gamma_2} \frac{1}{\{e^{(\pi/2\gamma_2)(x-b)} + e^{-(\pi/2\gamma_2)(x-b)}\}^2} \\ &= \frac{\pi^2}{2\gamma_2} \operatorname{sech}^2 \left\{ \frac{\pi}{2\gamma_2} (x-b) \right\} \\ &= \frac{\pi^2}{2\sqrt{3}\sigma} \operatorname{sech}^2 \left\{ \frac{\pi}{2\sqrt{3}\sigma} (x-b) \right\}, \end{aligned}$$

and the following expansion

$$(354) \quad f(x) = N \left\{ F(x) - \frac{a_4}{[4]} F^3(x) + \frac{a_5}{[5]} F^4(x) - \dots \right\},$$

where

$$F(x) = \frac{\pi}{4\sqrt{3}\sigma} \operatorname{sech}^2 \left\{ \frac{\pi}{2\sqrt{3}\sigma} (x-b) \right\}.$$

<sup>1</sup> Bierens de Haan. Loc. cit. Table 364, No. 1.

From (349) we obtain the following values of the successive coefficients:

$$\begin{aligned}
 (355) \quad & a_4 = 4\mu_3, \\
 & a_5 = 5\mu_4 - 21\sigma^4, \\
 & a_6 = 6\mu_5 - 60\sigma^2\mu_3, \\
 & a_7 = 7\mu_6 - 105\sigma^2\mu_4 + 162\sigma^6, \\
 & a_8 = 8\mu_7 - 168\sigma^2\mu_5 + 504\sigma^4\mu_3.
 \end{aligned}$$

We will now determine the coefficients both by Carver's method and by Charlier's method.

#### a. CARVER'S METHOD

The generating function, when the origin is at the mean, is

$$(356) \quad \frac{\kappa}{4} \operatorname{sech}^2 \frac{\kappa}{2} x, \quad \text{where} \quad \kappa = \frac{\pi}{\sqrt{3}\sigma}.$$

Employing the definite integral<sup>1</sup>

$$(357) \quad \int_{-\infty}^{\infty} \frac{x^{2a} dx}{(e^{qx} + e^{-qx})^2} = 2\pi^{2a} B_{2a-1} \frac{2^{2a-1} - 1}{(2q)^{2a+1}},$$

we get the following values:

$$\begin{aligned}
 (358) \quad & \mu_0 = 1, \\
 & \mu_2 = \frac{1}{3} \left( \frac{\pi}{\kappa} \right)^2, \\
 & \mu_4 = \frac{7}{15} \left( \frac{\pi}{\kappa} \right)^4, \\
 & \mu_6 = \frac{31}{21} \left( \frac{\pi}{\kappa} \right)^6, \\
 & \mu_1 = \mu_3 = \mu_5 = \dots = \mu_{2n+1} = 0.
 \end{aligned}$$

Simplifying,

$$(359) \quad \mu_2 = \sigma^2; \quad \mu_4 = \frac{21}{5} \sigma^4; \quad \mu_6 = \frac{279}{7} \sigma^6$$

and from these we obtain

$$\begin{aligned}
 (360) \quad & d_4 = \frac{21}{5} \sigma^4 - 3\sigma^4 = \frac{6}{5} \sigma^4, \\
 & d_6 = \frac{279}{7} \sigma^6 - 15\sigma^2 \cdot \frac{21}{5} \sigma^4 + 30\sigma^6 = \frac{48}{7} \sigma^6, \\
 & d_3 = d_5 = d_7 = \dots = d_{2n+1} = 0;
 \end{aligned}$$

<sup>1</sup> Bierens de Haan. Loc. cit. Table 86, No. 2.

whence

$$\begin{aligned}
 a_3 &= \mu_3, \\
 a_4 &= \mu_4 - 3\sigma^4 - \frac{6}{5}\sigma^4 = \frac{1}{5}(5\mu_4 - 21\sigma^4), \\
 (361) \quad a_5 &= \mu_5 - 10\sigma^2\mu_3, \\
 a_6 &= \mu_6 - 15\sigma^2\mu_4 + 30\sigma^6 - \frac{48}{7}\sigma^6 \\
 &= \frac{1}{7}(7\mu_6 - 105\sigma^2\mu_4 + 162\sigma^6).
 \end{aligned}$$

#### b. CHARLIER'S METHOD

We have, by definition,

$$\begin{aligned}
 (362) \quad \underline{s} \cdot \lambda_s &= \int_{-\infty}^{\infty} x^s F(x) dx = \frac{\pi}{\gamma_2} \int_{-\infty}^{\infty} \frac{x^s dx}{(e^{\kappa x/2} + e^{-(\kappa x/2)})^2} \\
 &= \frac{\pi}{\gamma_2} \int_{(s)} ,
 \end{aligned}$$

where

$$\kappa = \frac{\pi}{\sqrt{3}\sigma}.$$

From equation (357),

$$\begin{aligned}
 (363) \quad \frac{\pi}{\gamma_2} \int_{(0)} &= 1; & \frac{\pi}{\gamma_2} \int_{(2)} &= \frac{1}{3} \left( \frac{\pi}{\kappa} \right)^2; \\
 \frac{\pi}{\gamma_2} \int_{(4)} &= \frac{7}{15} \left( \frac{\pi}{\kappa} \right)^4; & \frac{\pi}{\gamma_2} \int_{(6)} &= \frac{31}{21} \left( \frac{\pi}{\kappa} \right)^6;
 \end{aligned}$$

therefore

$$\begin{aligned}
 (364) \quad \lambda_0 &= 1, \\
 \lambda_2 &= \frac{1}{3|2} \left( \frac{\pi}{\kappa} \right)^2 = \frac{1}{|2} \sigma^2, \\
 \lambda_4 &= \frac{7}{15|4} \left( \frac{\pi}{\kappa} \right)^4 = \frac{7}{40} \sigma^4, \\
 \lambda_6 &= \frac{31}{21|6} \left( \frac{\pi}{\kappa} \right)^6 = \frac{31}{560} \sigma^6, \\
 \lambda_1 &= \lambda_3 = \lambda_5 = \dots = \lambda_{2n+1} = 0,
 \end{aligned}$$

and, accordingly,

$$\begin{aligned}
 S_0(x) &= 1, \\
 S_1(x) &= -x, \\
 S_2(x) &= \frac{x^2}{\underline{2}} - \frac{1}{2}\sigma^2, \\
 S_3(x) &= -\frac{x^3}{\underline{3}} + \frac{x}{2}\sigma^2, \\
 S_4(x) &= \frac{x^4}{\underline{4}} - \frac{x^2}{4}\sigma^2 + \frac{3}{40}\sigma^4, \\
 S_5(x) &= -\frac{x^5}{\underline{5}} + \frac{x^3}{2\underline{3}}\sigma^2 - \frac{3}{40}x\sigma^4, \\
 S_6(x) &= \frac{x^6}{\underline{6}} - \frac{x^4}{2\underline{4}}\sigma^2 + \frac{3}{80}x^2\sigma^4 - \frac{3}{560}\sigma^6.
 \end{aligned}
 \tag{365}$$

From these the following values of the coefficients are readily obtained:

$$\begin{aligned}
 \underline{4} \cdot A_3 &= -4\mu_3, \\
 \underline{5} \cdot A_4 &= 5\mu_4 - 21\sigma^4, \\
 \underline{6} \cdot A_5 &= -6\mu_5 + 60\sigma^2\mu_3, \\
 \underline{7} \cdot A_6 &= 7\mu_6 - 105\sigma^2\mu_4 + 162\sigma^6.
 \end{aligned}
 \tag{366}$$

We will now find the successive derivatives of

$$\varphi(x) \equiv \operatorname{sech}^2 \kappa_1(x - b).$$

$$\begin{aligned}
 \varphi'(x) &= -2\kappa_1 \{ \tanh t \} \varphi(x); \quad t = \kappa_1(x - b), \\
 \varphi^2(x) &= 2\kappa_1^2 \{ 3 \tanh^2 t - 1 \} \varphi(x), \\
 \varphi^3(x) &= -8\kappa_1^3 \{ 3 \tanh^3 t - 2 \tanh t \} \varphi(x), \\
 \varphi^4(x) &= 8\kappa_1^4 \{ 15 \tanh^4 t - 15 \tanh^2 t + 2 \} \varphi(x),
 \end{aligned}
 \tag{367}$$

or, in general,

$$\begin{aligned}
 \varphi^{2n}(x) &= \kappa_1^{2n} \left\{ \sum_{j=0}^n A_{2n, 2j} \tanh^{2j} t \right\} \varphi(x); \\
 \varphi^{2n+1}(x) &= \kappa_1^{2n+1} \left\{ \sum_{j=0}^n A_{2n+1, 2j+1} \tanh^{2j+1} t \right\} \varphi(x).
 \end{aligned}
 \tag{368}$$

The  $A$ -coefficients of the  $(r+1)$ th derivative are derived from the  $A$ -coefficients of the  $r$ th derivative by the use of the



recurring formula

$$(369) \quad A_{r+1, s} = \{A_{r, s+1} - A_{r, s-1}\}(s+1);$$

provided we understand that

$$A_{r, -1} = 0,$$

$$A_{r, s} = 0, \text{ when } s > r.$$

A table similar to Table 6 may be constructed. The first six lines of such a table are given below.

TABLE 7

TABLE TO FACILITATE THE DETERMINATION OF THE  $r$ TH DERIVATIVE OF  $\text{SECH}^2 t$

|     | $s$  |     |      |            |            |     |      |     |     |
|-----|------|-----|------|------------|------------|-----|------|-----|-----|
|     | (-1) | (0) | (1)  | (2)        | (3)        | (4) | (5)  | (6) | (7) |
|     | 0    | 1   | 0    | 0          | 0          | 0   | 0    | 0   | 0   |
|     | 0    | 0   | -2   | 0          | 0          | 0   | 0    | 0   | 0   |
|     | 0    | -2  | 0    | 6          | 0          | 0   | 0    | 0   | 0   |
| $r$ | 0    | 0   | 16   | 0          | -24        | 0   | 0    | 0   | 0   |
|     | 0    | 16  | 0    | -120       | 0          | 120 | 0    | 0   | 0   |
|     | 0    | 0   | -272 | 0          | 960        | 0   | -720 | 0   | 0   |
|     |      |     |      | $A_{6, 2}$ |            |     |      |     |     |
|     |      |     |      |            | $A_{7, 3}$ |     |      |     |     |

Table 7 may be extended by the use of equation (369).

Substituting in (354) the values of the third, the fourth and the fifth derivatives, respectively, we obtain the following equation:

$$\begin{aligned}
 f(x) = F(x) \bigg\{ & 1 + \frac{a_4}{4} (8\kappa_1^3) [3 \tanh^3 t - 2 \tanh t] \\
 & + \frac{a_5}{5} (8\kappa_1^4) [15 \tanh^4 t - 15 \tanh^2 t + 2] \\
 & + \frac{a_6}{6} (16\kappa_1^5) [45 \tanh^5 t - 60 \tanh^3 t \\
 & \quad + 17 \tanh t] + \dots \bigg\},
 \end{aligned}
 \tag{370}$$

where

$$\begin{aligned}\kappa_1 &= \frac{\pi}{2\sqrt{3}\sigma}, \\ t &= \kappa_1(x - b), \\ F(x) &= \frac{\pi}{4\sqrt{3}\sigma} \left\{ \frac{\pi}{2\sqrt{3}\sigma} (x - b) \right\}.\end{aligned}$$

## CHAPTER FOUR

At this point, the question naturally arises: Is it possible to use as generating functions some powers of the hyperbolic secant other than the first and the second? To this question we have found a positive answer. In fact, we shall here introduce a family of generating functions of which the hyperbolic secant and the hyperbolic secant squared are but two of the individual members.

Let

$$(401) \quad {}_mI_{2n} = \int_{-\infty}^{\infty} x^{2n} \operatorname{sech}^m \gamma x dx,$$

such that for a chosen value of  $m$  we have the following relation:

$$\mu_{2n} \equiv \frac{{}_mI_{2n}}{{}_mI_0}.$$

By integration by parts, we get the following recurring formulas:

$$(402) \quad \begin{aligned} {}_mI_0 &= \frac{m-2}{m-1} \cdot {}_{m-2}I_0. \\ {}_mI_{2n} &= \frac{m-2}{m-1} \cdot {}_{m-2}I_{2n} - \frac{(2n-1)2n}{(m-2)(m-1)\gamma^2} \cdot {}_{m-2}I_{2n-2}. \end{aligned}$$

With these we are able to evaluate the even moments of any positive integral power of the hyperbolic secant. The odd moments, of course, vanish since an integral power of the hyperbolic secant is in general an even function.

### CASE I. EVEN POWERS OF THE HYPERBOLIC SECANT

We have, from the second recurring formula,

$${}_{2m}I_2 = \frac{2m-2}{2m-1} \cdot {}_{2m-2}I_2 - \frac{2}{(2m-1)(2m-2)\gamma^2} \cdot {}_{2m-2}I_0;$$

$m \leq 2.$

Similarly,

$${}_{2m-2}I_2 = \frac{2m-4}{2m-3} \cdot {}_{2m-4}I_2 - \frac{2}{(2m-3)(2m-4)\gamma^2} \cdot {}_{2m-4}I_0,$$

so that

$$\begin{aligned} {}_{2m}I_2 &= \frac{(2m-2)(2m-4)}{(2m-1)(2m-3)} \cdot {}_{2m-4}I_2 \\ &\quad - \frac{2m-2}{2m-1} \cdot \frac{2}{(2m-3)(2m-4)\gamma^2} \cdot {}_{2m-4}I_0 \\ &\quad - \frac{2}{(2m-1)(2m-2)\gamma^2} \cdot {}_{2m-2}I_0. \end{aligned}$$

Continuing this process repeatedly, we obtain the following expression for  ${}_{2m}I_2$ :

$$\begin{aligned} (403) \quad {}_{2m}I_2 &= \frac{(2m-2)(2m-4) \cdots 2}{(2m-1)(2m-3) \cdots 3} \cdot {}_2I_2 \\ &\quad - \frac{(2m-2)(2m-4) \cdots 4}{(2m-1)(2m-3) \cdots 5} \cdot \frac{2}{3 \cdot 2\gamma^2} \cdot {}_2I_0 \\ &\quad - \frac{2}{\gamma^2} \sum_{s=2}^{m-2} \frac{(2m-2)(2m-4) \cdots (2m-2s+2)}{(2m-1)(2m-3) \cdots (2m-2s+3)} \\ &\quad \quad \times \frac{1}{(2m-2s+1)(2m-2s)} \cdot {}_{2m-2s}I_0 \\ &\quad - \frac{2}{(2m-1)(2m-2)\gamma^2} \cdot {}_{2m-2}I_0. \end{aligned}$$

But it is easily seen from the first recurring formula that

$$(404) \quad {}_{2m}I_0 = \pi_{2m} \cdot {}_2I_0,$$

where

$$\pi_{2m} \equiv \frac{(2m-2)(2m-4) \cdots 2}{(2m-1)(2m-3) \cdots 3}.$$

Substituting in (403), we have

$$\begin{aligned} (405) \quad {}_{2m}I_2 &= \pi_{2m} \cdot {}_2I_2 \\ &\quad - \frac{2}{\gamma^2} \cdot {}_2I_0 \sum_{s=2}^{m-2} \frac{(2m-2)(2m-4) \cdots (2m-2s+2)}{(2m-1)(2m-3) \cdots (2m-2s+3)} \\ &\quad \quad \times \frac{1}{(2m-2s+1)(2m-2s)} \cdot \pi_{2m-2s} \\ &\quad - \frac{2}{(2m-1)(2m-2)\gamma^2} \cdot \pi_{2m-2} \cdot {}_2I_0 \\ &\quad - \frac{2}{2^2\gamma^2} \pi_{2m} \cdot {}_2I_0. \end{aligned}$$

or, simplifying,

$$(406) \quad {}_{2m}I_2 = \pi_{2m} \cdot {}_2I_2 - \frac{2}{\gamma^2} \cdot \pi_{2m} \cdot {}_2I_0 \sum_{s=2}^{m-1} \frac{1}{(2m-2s)^2} \\ - \frac{2}{\gamma^2} \cdot \pi_{2m} \cdot {}_2I_0 \frac{1}{(2m-2)^2} - \frac{2}{\gamma^2} \cdot \pi_{2m} \cdot {}_2I_0 \cdot \frac{1}{2^2}.$$

That is,

$$(407) \quad {}_{2m}I_2 = \pi_{2m} \left\{ {}_2I_2 - \frac{{}_2I_0}{2\gamma^2} \sum_{s=1}^{m-1} \frac{1}{s^2} \right\}.$$

From the second recurring formula, we deduce the following expression for  ${}_{2m}I_4$ :

$$\begin{aligned} {}_{2m}I_4 &= \frac{2m-2}{2m-1} \cdot {}_{2m-2}I_4 - \frac{12}{(2m-1)(2m-2)\gamma^2} \cdot {}_{2m-2}I_2 \\ &= \frac{(2m-2)(2m-4)}{(2m-1)(2m-3)} \cdot {}_{2m-4}I_4 \\ &\quad - \frac{2m-2}{2m-1} \frac{12}{(2m-3)(2m-4)\gamma^2} \cdot {}_{2m-4}I_2 \\ &\quad - \frac{12}{(2m-1)(2m-2)\gamma^2} \cdot {}_{2m-2}I_2. \end{aligned}$$

After successive application of this recurring relation, we obtain

$$(408) \quad \begin{aligned} {}_{2m}I_4 &= \pi_{2m} \cdot {}_2I_4 \\ &\quad - \frac{12}{\gamma^2} \sum_{s=2}^{m-2} \frac{(2m-2)(2m-4) \cdots (2m-2s+2)}{(2m-1)(2m-3) \cdots (2m-2s+3)} \\ &\quad \quad \times \frac{1}{(2m-2s+1)(2m-2s)} \cdot {}_{2m-2s}I_2 \\ &\quad - \frac{12}{(2m-1)(2m-3)\gamma^2} \cdot {}_{2m-2}I_2 \\ &\quad - \frac{(2m-2)(2m-4) \cdots 4}{(2m-1)(2m-3) \cdots 5} \cdot \frac{12}{3 \cdot 2\gamma^2} \cdot {}_2I_2. \end{aligned}$$

But, from (407),

$${}_{2m-2s}I_2 = \pi_{2m-2s} \left\{ {}_2I_2 - \frac{{}_2I_0}{2\gamma^2} \sum_{r=1}^{m-s-1} \frac{1}{r^2} \right\};$$

therefore the second term of the right-hand member of equa-

tion (408) becomes

$$\frac{12}{\gamma^2} \cdot \pi_{2m} \sum_{s=2}^{m-2} \frac{1}{(2m-2s)^2} \left\{ {}_2I_2 - \frac{{}_2I_0}{2\gamma^2} \sum_{r=1}^{m-s-1} \frac{1}{r^2} \right\},$$

and

$$\begin{aligned} & \frac{12}{(2m-1)(2m-3)\gamma^2} \cdot {}_2m-2I_2 \\ &= \frac{12}{(2m-2)^2\gamma^2} \cdot \pi_{2m} \left\{ {}_2I_2 - \frac{{}_2I_0}{2\gamma^2} \sum_{r=1}^{m-2} \frac{1}{r^2} \right\}. \end{aligned}$$

Equation (408) may now be reduced to the following form:

$$\begin{aligned} (409) \quad {}_{2m}I_4 &= \pi_{2m} \left\{ {}_2I_4 - \frac{3}{\gamma^2} \cdot {}_2I_2 - \frac{3}{\gamma^2} \sum_{s=1}^{m-2} \frac{1}{(m-s)^2} \left[ {}_2I_2 \right. \right. \\ &\quad \left. \left. - \frac{{}_2I_0}{2\gamma^2} \sum_{r=1}^{m-s-1} \frac{1}{r^2} \right] \right\} \\ &= \pi_{2m} \left\{ {}_2I_4 - \frac{3}{\gamma^2} \cdot {}_2I_2 - \frac{3}{\gamma^2} \sum_{s=2}^{m-1} \frac{1}{s^2} \left[ {}_2I_2 \right. \right. \\ &\quad \left. \left. - \frac{{}_2I_0}{2\gamma^2} \sum_{r=1}^{s-1} \frac{1}{r^2} \right] \right\}; \end{aligned}$$

whence

$$(410) \quad {}_{2m}I_4 = \pi_{2m} \left\{ {}_2I_4 - \frac{3}{\gamma^2} \cdot {}_2I_2 \sum_{s=1}^{m-1} \frac{1}{s^2} + \frac{3}{2\gamma^4} {}_2I_0 \sum_{s=1}^{m-1} \frac{1}{s^2} \sum_{r=1}^{s-1} \frac{1}{r^2} \right\}.$$

But it may be easily shown that

$$(411) \quad \sum_{s=2}^{m-1} \frac{1}{s^2} \sum_{r=1}^{s-1} \frac{1}{r^2} = \frac{1}{2} \left( \sum_{s=1}^{m-1} \frac{1}{s^2} \right)^2 - \frac{1}{2} \sum_{s=1}^{m-1} \frac{1}{s^4}.$$

Making use of this relation in equation (410), we have

$$\begin{aligned} (412) \quad {}_{2m}I_4 &= \pi_{2m} \left\{ {}_2I_4 - \frac{3}{\gamma^2} \cdot {}_2I_2 \sum_{s=1}^{m-1} \frac{1}{s^2} \right. \\ &\quad \left. + \frac{3}{4\gamma^4} \cdot {}_2I_0 \left[ \left( \sum_{s=1}^{m-1} \frac{1}{s^2} \right)^2 - \sum_{s=1}^{m-1} \frac{1}{s^4} \right] \right\}. \end{aligned}$$

But, from (356),

$${}_2I_0 = \frac{2}{\gamma}$$

and, from (357),

$$\begin{aligned} {}_2I_2 &= \frac{\pi^2}{6\gamma^3}; & \pi &= 3.14159\dots, \\ {}_2I_4 &= \frac{7\pi^4}{120\gamma^5}; \end{aligned}$$



accordingly

$$(413) \quad \begin{aligned} \frac{{}_2I_2}{{}_2I_0} &= \frac{\pi^2}{12\gamma^2}, \\ \frac{{}_2I_4}{{}_2I_0} &= \frac{7\pi^4}{240\gamma^4}, \end{aligned}$$

and, finally, the following values of the second and the fourth moments, respectively, of the hyperbolic secant raised to the  $2m$ th power:

$$(414) \quad \begin{aligned} {}_{2m}\mu_2 &= \frac{\pi^2}{12\gamma^2} - \frac{1}{2\gamma^2} \sum_{s=1}^{m-1} \frac{1}{s^2}, \\ {}_{2m}\mu_4 &= \frac{7\pi^4}{240\gamma^4} - \frac{\pi^2}{4\gamma^4} \sum_{s=1}^{m-1} \frac{1}{s^2} + \frac{3}{4\gamma^4} \left[ \left( \sum_{s=1}^{m-1} \frac{1}{s^2} \right)^2 - \sum_{s=1}^{m-1} \frac{1}{s^4} \right], \\ &\vdots \\ {}_{2m}\mu_1 &= {}_{2m}\mu_3 = {}_{2m}\mu_5 = \dots = {}_{2m}\mu_{2n+1} = 0. \end{aligned}$$

#### CASE II. ODD POWERS OF THE HYPERBOLIC SECANT

Proceeding in a similar manner as in Case I, we obtain the following equations:

$$(415) \quad \begin{aligned} {}_{2m+1}I_0 &= \pi {}_{2m+1}I_0, \\ {}_{2m+1}I_2 &= \pi {}_{2m+1} \left\{ {}_1I_2 - \frac{2}{\gamma^2} \cdot {}_1I_0 \sum_{s=0}^{m-1} \frac{1}{(2s+1)^2} \right\}, \\ {}_{2m+1}I_4 &= \pi {}_{2m+1} \left\{ {}_1I_4 - \frac{12}{\gamma^2} \cdot {}_1I_2 \sum_{s=0}^{m-1} \frac{1}{(2s+1)^2} \right. \\ &\quad \left. + \frac{24}{\gamma^4} \cdot {}_1I_0 \sum_{s=1}^{m-1} \frac{1}{(2s+1)^2} \sum_{r=0}^{s-1} \frac{1}{(2r+1)^2} \right\}, \end{aligned}$$

where

$$\pi {}_{2m+1} = \frac{(2m-1)(2m-3) \dots 1}{2m(2m-2) \dots 2}.$$

From the definite integral

$$(416) \quad \int_0^\infty \frac{x^{2a} dx}{e^{px} + e^{-px}} = \frac{1}{2} \left( \frac{\pi}{2p} \right)^{2a+1} \cdot B_{2a},$$

we get

$${}_1I_2 = \frac{\pi^3}{4\gamma^3},$$

$${}_1I_4 = \frac{5\pi^5}{16\gamma^5}.$$

Also, since

$${}_1I_0 = \frac{\pi}{\gamma}$$

it follows that

$$(417) \quad \begin{aligned} {}_{2m+1}\mu_2 &= \frac{\pi^2}{4\gamma^2} - \frac{2}{\gamma^2} \sum_{s=0}^{m-1} \frac{1}{(2s+1)^2} \\ {}_{2m+1}\mu_4 &= \frac{5\pi^4}{16\gamma^4} - \frac{3\pi^2}{\gamma^4} \sum_{s=0}^{m-1} \frac{1}{(2s+1)^2} \\ &\quad + \frac{24}{\gamma^4} \sum_{s=1}^{m-1} \frac{1}{(2s+1)^2} \sum_{r=0}^{s-1} \frac{1}{(2r+1)^2} \\ &\quad \vdots \\ {}_{2m+1}\mu_1 &= {}_{2m+1}\mu_3 = {}_{2m+1}\mu_5 = \cdots = {}_{2m+1}\mu_{2n+1} = 0. \end{aligned}$$

#### CONCLUDING REMARKS

With the successive moments evaluated, it is a simple matter to apply Carver's method of elimination and obtain the following series-expansion for the functional law:

$$(418) \quad f(x) = N \left\{ \theta(x) - \frac{A_3}{\underline{3}} \theta^3(x) + \frac{A_4}{\underline{4}} \theta^4(x) \right. \\ \left. - \frac{A_5}{\underline{5}} \theta^5(x) + \cdots \right\}.$$

These steps may be followed in graduating a frequency distribution by means of a power of the hyperbolic secant as a generating function.

1. *Preliminary Work.* Calculation and adjustment of moments around the mean. Determine  $\alpha_4$ , and refer to the reference table for the generalized-hyperbolic-secant curve, Table II, Appendix C, to choose which power of the hyperbolic secant to use.

2. *Evaluation of the Constants in the Generating Function so*

Chosen. If

$$\theta(x) = \kappa \operatorname{sech}^m \gamma x,$$

the constants to be evaluated are  $\kappa$  and  $\gamma$ . The reference table may be consulted again for this purpose.

3. *Evaluation of the Successive Coefficients.*

$$(419) \quad \begin{aligned} A_3 &= \mu_3, \\ A_4 &= (\Delta\alpha_4) \cdot \sigma^4, \\ A_5 &= \mu_5 - 10\sigma^2\mu_3, \end{aligned}$$

where  $\Delta\alpha_4$  stands for the difference between the  $\alpha_4$  of the observed distribution and the  $\alpha_4$  of the generating function.

4. *Calculation of the Different Columns of the Graduation Proper.* The following values of the second, the third, and the fourth derivatives, respectively, of

$$\theta(x) \equiv \operatorname{sech}^m \gamma x$$

are given for reference:

$$(420) \quad \begin{aligned} \frac{\theta^2(x)}{\theta(x)} &= \gamma^2 \{-m + m(m+1) \tanh^2 \gamma x\}, \\ \frac{\theta^3(x)}{\theta(x)} &= m\gamma^3 \{(3m+2) \tanh \gamma x \\ &\quad - (m+1)(m+2) \tanh^3 \gamma x\}, \\ \frac{\theta^4(x)}{\theta(x)} &= m\gamma^4 \{(3m+2) - 2(3m^2 + 7m + 4) \tanh^2 \gamma x \\ &\quad + (m+1)(m+2)(m+3) \tanh^4 \gamma x\}. \end{aligned}$$

We shall now take up a numerical example to illustrate these different steps.

#### FOURTH NUMERICAL EXAMPLE

The rough data is taken from *The Medical Department of the United States Army in the World War*, vol. XV, Part I, page 302. It gives the distribution of weights of 10701 recruits with pulmonary tuberculosis.

*A. Preliminary Work.* Table 8 gives the different steps in the preliminary work of evaluating the  $\bar{\alpha}_s$ . Attention is here called again to the new method of reducing the moments around the arbitrary origin to the corresponding moments

around the mean. The value of  $\alpha_4$  of the observed distribution is 3.69754. Turning to Table II, Appendix C, we find that this value is about halfway between the values of  ${}_3\alpha_4$  and  ${}_4\alpha_4$ . For our generating function, there is very little to choose then between  $\text{sech}^3 \gamma x$  and  $\text{sech}^4 \gamma x$ ; so we have graduated the observed data first by the use of  $\kappa \text{sech}^3 \gamma x$  as a generating function, and secondly by the use of  $\kappa \text{sech}^4 \gamma x$ . The first graduation is given in detail in the following pages, while only the final results of the second graduation are presented.

TABLE 8

TABLE SHOWING THE SUCCESSIVE STEPS IN CALCULATING THE ALPHAS OF AN OBSERVED DISTRIBUTION

*Fourth Numerical Example*

| <i>s</i> | Moment about Arbitrary Origin<br>$\nu_s^1$ | Powers of <i>b</i> , Distance of Mean from Arbitrary Origin<br>$b^s$ | Ratio of Column (2) to Column (3)<br>$\nu_s^1/b^s$ |
|----------|--------------------------------------------|----------------------------------------------------------------------|----------------------------------------------------|
| (1)      | (2)                                        | (3)                                                                  | (4)                                                |
| 1        | .6883468835                                | .6883468835                                                          | 1                                                  |
| 2        | 9.164564059                                | .4738214320                                                          | 19.34180989                                        |
| 3        | 32.99448650                                | .3261535061                                                          | 101.1624462                                        |
| 4        | 343.7650687                                | .2245067495                                                          | 1531.201487                                        |
| 5        | 2544.736567                                | .1545385213                                                          | 16466.68122                                        |
| 6        | 27605.09392                                | .1063761095                                                          | 259504.6392                                        |
| 7        | 286966.4399                                | .0732236635                                                          | 3919039.631                                        |
| 8        | 3426935.052                                | .0504032806                                                          | 67990317.59                                        |

*Calculation of Standard Deviation*

$$\begin{aligned}\mu_2 &= 8.607409295 \\ \sigma &= \sqrt{\mu_2} = 2.933838662 \\ \frac{1}{\sigma} &= .3408503722\end{aligned}$$

| <i>s</i> | Unadjusted Moment about Mean<br>$\nu_s = b^s \cdot \sum_{n=0}^s \frac{\nu_n}{b^n}$ | Adjusted Moment about Mean<br>$\mu_s$ | Powers of Standard Deviation<br>$\sigma^s$ | Observed Alpha<br>$\bar{\alpha}_s = \frac{\mu_s}{\sigma^s}$ |
|----------|------------------------------------------------------------------------------------|---------------------------------------|--------------------------------------------|-------------------------------------------------------------|
| (1)      | (5)                                                                                | (6)                                   | (7)                                        | (8)                                                         |
| 3        | 14.72159619                                                                        | 14.72159619                           | 25.25275017                                | .582970017                                                  |
| 4        | 278.2991423                                                                        | 273.94127099                          | 74.08749477                                | 3.697537249                                                 |
| 5        | 1488.651053                                                                        | 1476.383056                           | 217.3607565                                | 6.792316514                                                 |
| 6        | 19353.479241                                                                       | 19009.43653                           | 637.7013910                                | 29.80930698                                                 |
| 7        | 175579.79559                                                                       | 172989.68454                          | 1870.912996                                | 92.4627093                                                  |
| 8        | 2171577.4469                                                                       | 2146981.858                           | 5488.956881                                | 387.502016                                                  |

*B. Evaluation of Constants and of Successive Coefficients.*

With the use of Table 8 which we have just given and of Table II, Appendix C, we can evaluate at once the constants  $\kappa$  and  $\gamma$  and the successive coefficients  $A_3$ ,  $A_4$ , and  $A_5$ . But we shall show that a slight modification of the series would simplify the graduation of the observed data. This modification would also alter the form of the constants and the coefficients to be evaluated. Taking just the first four terms of the series, we have the following expression for our unknown law of distribution:

$$y = N \left\{ \theta(x) - \frac{A_3}{\underline{3}} \theta^3(x) + \frac{A_4}{\underline{4}} \theta^4(x) - \frac{A_5}{\underline{5}} \theta^5(x) \right\}.$$

Integrating both sides of the equation with respect to  $x$ , we have

$$(421) \quad \int_{-\infty}^x y dx = N \left\{ \frac{\kappa}{\gamma} \int_{-\infty}^t \varphi(t) dt - \frac{A_3}{\underline{3}} \theta^2(x) + \frac{A_4}{\underline{4}} \theta^3(x) - \frac{A_5}{\underline{5}} \theta^4(x) \right\},$$

where

$$\varphi(t) \equiv \operatorname{sech}^3 t.$$

It must be observed that we have tacitly assumed that the unknown law of distribution is adequately represented by the first four terms of the series; hence the integration term by term of the right-hand member of the equation is perfectly legitimate.

It may be easily shown that

$$(422) \quad \begin{aligned} -\frac{A_3}{\underline{3}} \theta^2(x) &= -\frac{A_3'}{\underline{3}} \left\{ \frac{\kappa}{\gamma} (\gamma\sigma)^3 \varphi^2(t) \right\}, \\ \frac{A_4}{\underline{4}} \theta^3(x) &= \frac{A_4'}{\underline{4}} \left\{ \frac{\kappa}{\gamma} (\gamma\sigma)^4 \varphi^3(t) \right\}, \\ -\frac{A_5}{\underline{5}} \theta^4(x) &= -\frac{A_5'}{\underline{5}} \left\{ \frac{\kappa}{\gamma} (\gamma\sigma)^5 \varphi^4(t) \right\}, \end{aligned}$$

where

$$\begin{aligned}
 (423) \quad A_3' &= \alpha_3, \\
 A_4' &= \Delta\alpha_4, \\
 A_5' &= \alpha_5 - 10\alpha_3.
 \end{aligned}$$

Now using Table 8 of this chapter and Table II, Appendix C, we get the following values of the different constants and coefficients we need for graduation:

$$\begin{aligned}
 (424) \quad \gamma &= .23303, \\
 \frac{\kappa}{\gamma} &= .6366198, \\
 \frac{A_3'}{\underline{3}} &= .09716167, \\
 \frac{A_4'}{\underline{4}} &= - .0045297, \\
 \frac{A_5'}{\underline{5}} &= .008021803.
 \end{aligned}$$

With these we are now ready to proceed with the graduation proper.

*C. Graduation Proper.* Table 9 gives the different columns of the graduation proper.

Column (1). Boundary points of the class intervals. Note that the values taken are the mean values of the natural integral numbers.

Column (2). There are two distinct steps involved here; the first is the transferring of the origin to the mean, and the second, the reducing of the new values of  $x$  to standard units. For convenience, we have designated the new unit by the letter " $t$ ."

Columns (3), (4), and (5). These three columns give the intermediate steps in evaluating the area of the curve generated by the first term of the series. We have

$$\theta(x) = \kappa \operatorname{sech}^3 \gamma x$$



from which it follows that

$$(425) \quad \int_{-\infty}^x \theta(x) dx = \frac{\kappa}{\gamma} \left[ \frac{1}{2} \operatorname{sech} t \tanh t + \frac{1}{2} \int_{-\infty}^t \operatorname{sech} t dt \right] \\ = \frac{1}{\pi} \operatorname{sech} t \tanh t + A(t).$$

The values of  $\tanh t$  and  $\operatorname{sech} t$  are from the Tables of Hyperbolic Functions published by the Smithsonian Institution and from Gudermann's tables found in Crelle's *JOURNAL FÜR DIE REINE UND ANGEWANDTE MATHEMATIK*, Bd. 8 und Bd. 9. The values of Column (5) are taken from Table V, Appendix C. Straight-line interpolation is employed.

Column (6). This gives the sum of Column (4) and Column (5), or the area bounded by the generating curve and the axis of reals between  $-\infty$  and  $t$ .

Columns (7), (8), and (9). These are the first, the second and the third correctional terms, respectively. These columns are built up by the use of the following formulae:

$$(426) \quad \begin{aligned} \text{Column (7)} &= -\frac{A_3'}{[3]} \left\{ \frac{\kappa}{\gamma} (\gamma\sigma)^3 \varphi^2(t) \right\}, \\ \text{Column (8)} &= \frac{A_4'}{[4]} \left\{ \frac{\kappa}{\gamma} (\gamma\sigma)^4 \varphi^3(t) \right\}, \\ \text{Column (9)} &= -\frac{A_5'}{[5]} \left\{ \frac{\kappa}{\gamma} (\gamma\sigma)^5 \varphi^4(t) \right\}. \end{aligned}$$

The remaining columns of Table 9 are self-explanatory.

*Note on Tables of Hyperbolic Functions:* For most purposes the tables of the Smithsonian Institution of Washington, D. C., are the best. They give in a well-arranged manner the values of  $\sinh$ ,  $\cosh$  and  $\tanh$  to five decimal places. The range of the argument is from 0.0000 to .1000, from .101 to 3.000, and from 3.01 to 6.00. Other tables are

PEIRNOT AND WOODS. *Logarithms of Hyperbolic Functions to Twelve Significant Figures*. Berkeley, 1918. The range of the argument is from .000 to 2.032 inclusive.

Gudermann's Tables. Published in Crelle's *JOURNAL FÜR DIE REINE UND ANGEWANDTE MATHEMATIK*, Bd. 8 und Bd. 9. The range is from 2.000 to 5.000, and from 5.01 to 12.00. These tables give the common logarithms of  $\sinh$ ,  $\cosh$  and  $\tanh$  to nine significant figures.



TABLE 9—(Continued)

| (1)  | (2)   | (3)     | (4)     | (5)     | (6)     | (7)     | (8)     | (9)     | (10)    | (11)    | (12)    | (13) | (14)    | (15)    | (16)    | (17)    |         |         |         |         |      |
|------|-------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|------|---------|---------|---------|---------|---------|---------|---------|---------|------|
| —    | 0.5   | —       | 0.277   | —       | 0.2609  | —       | 0.8279  | 0.41264 | 0.32985 | 0.3748  | 0.0435  | —    | 0.0827  | 0.36733 | 0.37168 | 0.36341 | 0.14217 | 0.16336 | 0.15992 | 0.14355 | —    |
| 0.5  | —     | 0.044   | —       | 0.0398  | 0.48600 | 0.47202 | —       | 0.0393  | 0.48600 | 0.5867  | 0.0091  | —    | 0.2464  | 0.53069 | 0.53160 | 0.50696 | 0.14619 | 0.13583 | 0.13153 | 0.13986 | 0.5  |
| 1.5  | 0.189 | 0.18349 | 0.5841  | 0.5980  | 0.61821 | 0.4837  | —       | 0.0345  | —       | 0.04837 | 0.0091  | —    | 0.0345  | 0.66658 | 0.66313 | 0.64682 | 0.12866 | 0.09696 | 0.9586  | 0.11637 | 1.5  |
| 2.5  | 0.422 | 0.36557 | 0.11636 | 0.63051 | 0.74687 | 0.01667 | —       | 0.0455  | 0.0040  | 0.76354 | 0.75899 | —    | 0.0440  | 0.76354 | 0.75899 | 0.76339 | 0.09789 | 0.07074 | 0.07268 | 0.08115 | 2.5  |
| 3.5  | 0.655 | 0.47045 | 0.14975 | 0.69501 | 0.84476 | —       | 0.01048 | —       | 0.00261 | 0.01048 | 0.00261 | —    | 0.01287 | 0.83428 | 0.83167 | 0.84454 | 0.06588 | 0.05530 | 0.05749 | 0.05333 | 3.5  |
| 4.5  | 0.888 | 0.49998 | 0.15915 | 0.75149 | 0.91064 | —       | 0.02106 | —       | 0.0042  | 0.02106 | 0.0042  | —    | 0.00871 | 0.88958 | 0.88916 | 0.89774 | 0.04031 | 0.04182 | 0.04288 | 0.03657 | 4.5  |
| 5.5  | 1.121 | 0.47610 | 0.15155 | 0.79940 | 0.95095 | —       | 0.01955 | —       | 0.0064  | 0.01955 | 0.0064  | —    | 0.00270 | 0.93140 | 0.93204 | 0.93474 | 0.02302 | 0.02869 | 0.02886 | 0.02571 | 5.5  |
| 6.5  | 1.354 | 0.42361 | 0.13484 | 0.83913 | 0.97397 | —       | 0.01388 | —       | 0.0081  | 0.01388 | 0.0081  | —    | 0.00445 | 0.96009 | 0.96090 | 0.96045 | 0.01254 | 0.01787 | 0.01768 | 0.01680 | 6.5  |
| 7.5  | 1.587 | 0.36112 | 0.11495 | 0.87156 | 0.98651 | —       | 0.00855 | —       | 0.0062  | 0.00855 | 0.0062  | —    | 0.0123  | 0.97796 | 0.97858 | 0.97725 | 0.00660 | 0.01029 | 0.01007 | 0.01035 | 7.5  |
| 8.5  | 1.820 | 0.29961 | 0.09537 | 0.89774 | 0.99311 | —       | 0.00486 | —       | 0.0040  | 0.00486 | 0.0040  | —    | 0.0103  | 0.98825 | 0.98865 | 0.98760 | 0.00340 | 0.00364 | 0.00547 | 0.00583 | 8.5  |
| 9.5  | 2.053 | 0.24435 | 0.07778 | 0.91873 | 0.99651 | —       | 0.00262 | —       | 0.0023  | 0.00262 | 0.0023  | —    | 0.0069  | 0.99389 | 0.99412 | 0.99343 | 0.0173  | 0.02098 | 0.0287  | 0.00316 | 9.5  |
| 10.5 | 2.286 | 0.19715 | 0.06275 | 0.93549 | 0.99824 | —       | 0.00137 | —       | 0.0012  | 0.00137 | 0.0012  | —    | 0.0040  | 0.99687 | 0.99699 | 0.99659 | 0.00089 | 0.0156  | 0.0150  | 0.01068 | 10.5 |
| 11.5 | 2.519 | 0.15798 | 0.05029 | 0.94884 | 0.99913 | —       | 0.0070  | —       | 0.0006  | 0.0070  | 0.0006  | —    | 0.0022  | 0.99843 | 0.99849 | 0.99827 | 0.00043 | 0.00077 | 0.0074  | 0.00084 | 11.5 |
| 12.5 | 2.752 | 0.12605 | 0.04012 | 0.95944 | 0.99956 | —       | 0.0036  | —       | 0.0003  | 0.0036  | 0.0003  | —    | 0.0012  | 0.99920 | 0.99923 | 0.99911 | 0.00022 | 0.00040 | 0.00039 | 0.00045 | 12.5 |
| 13.5 | 2.985 | 0.10031 | 0.03193 | 0.96785 | 0.99978 | —       | 0.0018  | —       | 0.0002  | 0.0018  | 0.0002  | —    | 0.0006  | 0.99960 | 0.99962 | 0.99956 | 0.00011 | 0.00020 | 0.00019 | 0.00022 | 13.5 |
| 14.5 | 3.219 | 0.07961 | 0.02534 | 0.97455 | 0.99989 | —       | 0.0009  | —       | 0.0001  | 0.0009  | 0.0001  | —    | 0.0003  | 0.99980 | 0.99981 | 0.99978 | 0.0006  | 0.00011 | 0.00010 | 0.00011 | 14.5 |
| 15.5 | 3.452 | 0.06317 | 0.02011 | 0.97984 | 0.99995 | —       | 0.0004  | —       | 0.0000  | 0.0004  | 0.0000  | —    | 0.0002  | 0.99991 | 0.99991 | 0.99989 | 0.0003  | 0.00005 | 0.00005 | 0.00006 | 15.5 |
| 16.5 | 3.685 | 0.05010 | 0.01595 | 0.98403 | 0.99998 | —       | 0.0002  | —       | 0.0000  | 0.0002  | 0.0000  | —    | 0.0001  | 0.99996 | 0.99996 | 0.99995 | 0.00003 | 0.00005 | 0.00005 | 0.00006 | 16.5 |

*D. Final Results.* The final results are given in Table 10. Column (2) gives the rough table, or the observed frequency distribution. Four different theoretical distributions are given in Columns (3), (4), (5), and (6). These are obtained by multiplying Columns (13), (14), (15), and (16), respectively, of Table 9 by 10701, the total frequency.

TABLE 10

DISTRIBUTION OF WEIGHTS OF 10701 RECRUITS WITH PULMONARY TUBERCULOSIS

*Fourth Numerical Example*

*Source of Rough Data:* *The Med. Dept. of the U. S. Army*, vol. XV, Part I, p. 302.

*Generating Function Employed:*  $\varphi(t) = \kappa \operatorname{sech}^3 t$ .

| Class Interval | Observed Frequency | Graduation    |                |                  |                 |
|----------------|--------------------|---------------|----------------|------------------|-----------------|
|                |                    | One Term Used | Two Terms Used | Three Terms Used | Four Terms Used |
|                |                    | First         | Second         | Third            | Fourth          |
| (1)            | (2)                | (3)           | (4)            | (5)              | (6)             |
| Pounds         |                    |               |                |                  |                 |
| 89 and under   | 3                  | 56            | 16             | 11               | 3               |
| 90-94          | 9                  | 55            | 22             | 20               | 17              |
| 95-99          | 46                 | 106           | 55             | 52               | 55              |
| 100-104        | 167                | 197           | 134            | 134              | 156             |
| 105-109        | 372                | 352           | 310            | 317              | 372             |
| 110-114        | 718                | 591           | 645            | 665              | 728             |
| 115-119        | 1186               | 914           | 1140           | 1165             | 1137            |
| 120-124        | 1462               | 1259          | 1609           | 1612             | 1421            |
| 125-129        | 1498               | 1521          | 1748           | 1711             | 1536            |
| 130-134        | 1460               | 1564          | 1454           | 1408             | 1497            |
| 135-139        | 1142               | 1377          | 1038           | 1026             | 1247            |
| 140-144        | 913                | 1048          | 757            | 778              | 868             |
| 145-149        | 642                | 705           | 592            | 615              | 571             |
| 150-154        | 435                | 431           | 448            | 459              | 395             |
| 155-159        | 235                | 246           | 307            | 309              | 275             |
| 160-164        | 167                | 134           | 191            | 189              | 180             |
| 165-169        | 133                | 71            | 110            | 108              | 111             |
| 170-174        | 47                 | 36            | 60             | 59               | 62              |
| 175-179        | 29                 | 19            | 32             | 31               | 34              |
| 180-184        | 13                 | 10            | 17             | 16               | 18              |
| 185-189        | 9                  | 5             | 8              | 8                | 9               |
| 190-194        | 5                  | 2             | 4              | 4                | 5               |
| 195-199        | 8                  | 1             | 2              | 2                | 2               |
| 200-204        | 2                  | 1             | 2*             | 2*               | 2*              |

\* Includes area beyond the interval 200-204.

*Fifth Numerical Example*

Table 11 gives the final results of graduating the rough data used in the preceding example by means of the fourth power of the hyperbolic secant as the generating function.

TABLE 11

DISTRIBUTION OF WEIGHTS OF 10701 RECRUITS WITH PULMONARY TUBERCULOSIS

*Fifth Numerical Example*

Source of Rough Data: *The Med. Dept. of U. S. Army*, vol. XV, Part I, p. 302.

Generating Function Employed:  $\varphi(t) = \kappa \operatorname{sech}^4 t$ .

| Class Interval | Observed Frequency | Graduation    |                |                  |                 |
|----------------|--------------------|---------------|----------------|------------------|-----------------|
|                |                    | One Term Used | Two Terms Used | Three Terms Used | Four Terms Used |
|                |                    | First         | Second         | Third            | Fourth          |
| (1)            | (2)                | (3)           | (4)            | (5)              | (6)             |
| Pounds         |                    |               |                |                  |                 |
| 89 and under   | 3                  | 50            | 6              | 10               | 3               |
| 90-94          | 9                  | 54            | 18             | 20               | 18              |
| 95-99          | 46                 | 106           | 54             | 56               | 63              |
| 100-104        | 167                | 203           | 145            | 144              | 170             |
| 105-109        | 372                | 363           | 337            | 329              | 380             |
| 110-114        | 718                | 610           | 684            | 667              | 707             |
| 115-119        | 1186               | 923           | 1144           | 1126             | 1080            |
| 120-124        | 1462               | 1258          | 1561           | 1561             | 1407            |
| 125-129        | 1498               | 1495          | 1678           | 1707             | 1585            |
| 130-134        | 1460               | 1528          | 1440           | 1476             | 1537            |
| 135-139        | 1142               | 1364          | 1080           | 1092             | 1258            |
| 140-144        | 913                | 1050          | 782            | 769              | 860             |
| 145-149        | 642                | 723           | 594            | 574              | 556             |
| 150-154        | 435                | 445           | 441            | 429              | 376             |
| 155-159        | 235                | 255           | 307            | 304              | 267             |
| 160-164        | 167                | 136           | 193            | 194              | 181             |
| 165-169        | 133                | 70            | 113            | 115              | 115             |
| 170-174        | 47                 | 35            | 61             | 63               | 66              |
| 175-179        | 29                 | 17            | 31             | 33               | 36              |
| 180-184        | 13                 | 8             | 16             | 16               | 19              |
| 185-189        | 9                  | 4             | 8              | 8                | 9               |
| 190-194        | 5                  | 2             | 4              | 4                | 4               |
| 195-199        | 8                  | 2             | 2              | 2                | 2               |
| 200-204        | 2                  | 0             | 2              | 2                | 2               |

The following are the values of the constants and the coefficients used in this graduation:

$$\begin{aligned}
 N &= 10701., \\
 \frac{\kappa}{\gamma} &= .7500000, \\
 \gamma &= .19356, \\
 (427) \quad \frac{A_3'}{\underline{3}} &= .09716167, \\
 \frac{A_4'}{\underline{4}} &= - .0045297, \\
 \frac{A_5'}{\underline{5}} &= .008021803.
 \end{aligned}$$



## CHAPTER FIVE

In the preceding pages we have tried to show the possibility of developing the theory of graduation of frequency distributions as a problem in pure mathematics. We started with a form of Fourier's double integral and deduced not only Charlier's Type A curve but a number of other expansions as well. We have shown at least that there seems to exist a connection between the subject of frequency distributions and that of Fourier's integrals. Just how close this connection is we have not determined. It would be a great forward step if it could be proved conclusively that Fourier's series and Fourier's integrals may be used in graduating certain classes of frequency distributions. This would mean that all the existing theory and facts related to this subject would be available for use in statistical mathematics.

In the course of the preparation of this paper, a number of questions have presented themselves. These problems are enumerated in the following pages. Some of them are concerned with pure analysis which might interest a mathematician even though his field may not be mathematical statistics. First and foremost is the question of convergence. We have arrived at the following series-expansion for the unknown law of distribution:

$$(501) \quad f(x) = N \left\{ \varphi(t) - \frac{A_3}{1 \cdot 3} \varphi^3(t) + \frac{A_4}{1 \cdot 4} \varphi^4(t) - \dots \right\},$$

where  $\varphi(t)$  may be any one of the generating functions we have found. Now the question may be raised: Under what conditions is the series convergent? If it does converge under certain conditions, how rapid is the convergence? How significant are the terms after, say the third or the fourth?

Obviously the convergence of this series depends upon the nature of the successive coefficients  $A_3, A_4, \dots, A_n$ , and these, it must be noted, are functions of the unknown functional

law of the frequency distribution. It is evident, therefore, that even for a particular form of  $\varphi(t)$  the problem of finding a general test for convergence of the series in (501) is not easy.

Closely related to the foregoing question is the question of "where to stop." We have an unknown function expressed as a series. How many terms of the series must we employ in evaluating the unknown function. In pure mathematics this question is more easily answered for there the degree of accuracy we desire to attain in the final results determines the number of terms we have to take. But in statistics the question becomes more complicated on account of the fact that the greater the number of terms we include, the higher the degree of the moments or of the alphas involved in the coefficients. Which means that the question of probable errors has to be taken into consideration besides the degree of accuracy desired. To carry the contrast still further, in pure mathematics different ranges of the argument require different numbers of terms of the series to be employed. Let us take a specific example. In the calculation of Table V, Appendix C, the first three terms of the determining formula (see equation (A19), page 124, Appendix C) are sufficient for evaluating  $A(-t)$  for values of  $t$  between .00 and .06 inclusive. When  $t$  is equal to .35, eight terms are necessary, while when  $t$  is equal to .80, no less than ten terms of the series are needed in order to attain the same degree of accuracy. Now the question arises: Would it be necessary in the graduation of a frequency distribution to take different numbers of terms of the series in (501) for different ranges of the variate and thus secure a uniform degree of accuracy for the graduated distribution? Certainly it seems reasonable to suppose that in the first numerical example, chapter one, for instance, it would take a smaller number of terms to calculate the values at the extremities of the table than those values which are clustered around the mean.

This difficulty can be avoided if we look at the problem of graduation from a slightly different angle. We may consider any observed frequency table as a faulty representation of an

unknown frequency distribution. From this observed table with all the errors incorporated into it, we propose to find the true frequency distribution. For a first approximation we form a theoretical distribution of values found by using just the first term of the series. Then for a second approximation we employ the first two terms of the series as the functional law. Then we may form another theoretical distribution by taking the first three terms of the series as the functional law. And so on. Then choose that frequency distribution which would fit most closely the observed data. In other words, decide the question of "where to stop" for the distribution as a whole, and not for each individual frequency.

But how are we to know which of the theoretical distributions would fit the observed data most closely? A number of answers have already been given by various statisticians. We suggest the following criterion which we believe is more practicable than, and at least as reasonable as, any other criterion we know of. It is rather unique in that it could decide the question of good fit before any graduation is actually effected.

If we take two sets of statistical items and arrange each set into a frequency table according to some common class characteristic, the two resulting distributions would not be absolutely identical. If a third table is constructed from another set of data, there would result a third distribution which would be identical to neither of the other two. Thus the three different sets would give rise to three frequency distributions which, though possessing the same general form, exhibit slight variations, however homogeneous may be the class from which the individual items of the sets have been taken.

In the first example, chapter one, we have the distribution of heights of 6441 colored soldiers. It would be unreasonable to suppose that if we take another group of 6441 colored soldiers and record their heights, the resulting frequency table would be absolutely identical to the one given here. Different sets lead to different frequency distributions, although the variations of the distributions may be slight as we go from

one set to another. To predict the exact form of the frequency table which a certain set would give rise to is therefore an impossible task. But we can assume that there exists a limiting form which the frequency distribution approaches when we increase the number of items in the set to infinity. The problem of graduation would then reduce to finding the functional law of this limiting form. With such a law we could build up a theoretical distribution and say that this is representative of the class as a whole. That an observed frequency table would be found to be an exact reproduction of this theoretical distribution is highly improbable. A distribution of a finite set will in general exhibit slight variations from the theoretical distribution. But these variations ought to be insignificant. Conversely, if after deducing a functional law and building up a theoretical distribution from this law we note that the variations of the observed table from which we have deduced the functional law are reasonable or insignificant, then we say that the functional law deduced is correct.

This statement requires a limitation to the term "reasonable variations." Different definitions of this term would give rise to different criteria of good fit. For our purposes we shall consider a variation of a quantity reasonable if it does not exceed in absolute value the probable error of the quantity. Using algebraic symbolism, we may say that a variation is reasonable, or what is the same, insignificant, if

$$(502) \quad |\epsilon| \equiv |T - O| \not> \text{P.E.},$$

where

$\epsilon$  = variation,

$T$  = theoretical value,

$O$  = observed value,

P.E. = probable error of the quantity.

The phrase "variations of the distribution" is also capable of a number of different interpretations. In our discussion we shall use it to mean the variations of the theoretical alphas, which, it must be recalled, are defined in the following

manner:

$$\alpha_s \equiv \frac{\mu_s}{\sigma^s}.$$

Obviously such an interpretation is perfectly justifiable.

There are three distinct steps in the practical application of this criterion. In the first place we have to evaluate the theoretical alphas of the graduated distribution. Then their corresponding probable errors must be calculated. Lastly, we have to determine if the differences between the theoretical alphas and the observed alphas are reasonable, that is, not greater than their corresponding probable errors. If we let

$\alpha_s$  = the  $s$ th alpha of the theoretical distribution,

$\bar{\alpha}_s$  = the  $s$ th alpha of the observed distribution,

$\alpha_s'$  = the  $s$ th alpha of the generating function,

we may write equation (502) thus:

$$(503) \quad |\epsilon| \equiv |\alpha_s - \bar{\alpha}_s| \succcurlyeq \text{P.E.}$$

or

$$(504) \quad |\epsilon| \equiv |U(\alpha_s') - \bar{\alpha}_s| \succcurlyeq \text{P.E.},$$

since  $\alpha_s$  is a function of the alphas of the generating function.

The observed alphas are evaluated directly from the original data as has been shown previously. The theoretical alphas are functions of the alphas of the generating function and may be defined as follows:

$$(505) \quad \alpha_n = \alpha_n' + |n| \cdot \sum_{i=3}^{n:r} \frac{1}{|i| |n-i|} \cdot \frac{A_i}{\sigma^i} \cdot \alpha_{n-i}'; \quad n \leq 3.$$

The superior limit of the finite summation is the smaller of the two quantities  $n$  and  $r$ . For a proof of the preceding equation, we need the following lemma:

*Lemma:* If

$$(506) \quad y = \varphi(x) - \frac{A_3}{|3|} \varphi^3(x) + \frac{A_4}{|4|} \varphi^4(x) - \dots + (-1)^r \frac{A_r}{|r|} \varphi^r(x),$$

then

$$\mu_0 = \mu_0'; \quad \mu_1 = \mu_1'; \quad \mu_2 = \mu_2'$$

and

$$(507) \quad \mu_n = \mu_n' + \lfloor n \cdot \sum_{i=3}^{n:r} \frac{A_i}{i \lfloor n-i \rfloor} \mu_{n-i}' ; \quad n \nless 3,$$

where

$$\mu_n' \equiv \int_{-\infty}^{\infty} x^n \varphi(x) dx.$$

The smaller of the two quantities  $n$  and  $r$  is taken as the superior limit of the finite summation in (507).

*Proof:* Multiply both sides of equation (506) by  $x^n$  and integrate with respect to  $x$  between the limits  $-\infty$  and  $+\infty$ :

$$(508) \quad \begin{aligned} \mu_n = \mu_n' - \frac{A_3}{\lfloor 3 \rfloor} \int_{-\infty}^{\infty} x^n \varphi^3(x) dx + \frac{A_4}{\lfloor 4 \rfloor} \int_{-\infty}^{\infty} x^n \varphi^4(x) dx \\ + \dots + (-1)^r \frac{A_r}{\lfloor r \rfloor} \int_{-\infty}^{\infty} x^n \varphi^r(x) dx. \end{aligned}$$

But, by successive integration by parts, we have the following reduction formulas:

$$(509) \quad \begin{aligned} \int_{-\infty}^{\infty} x^s \varphi^r(x) dx &= (-1)^r s^{(r)} \mu_{s-r}' ; & (s-r) \nless 0, \\ \int_{-\infty}^{\infty} x^s \varphi^r(x) dx &= 0 ; & (s-r) < 0. \end{aligned}$$

whence

$$\mu_0 = \mu_0'; \quad \mu_1 = \mu_1'; \quad \mu_2 = \mu_2';$$

and

$$(510) \quad \begin{aligned} \mu_n = \mu_n' + \frac{A_3}{\lfloor 3 \rfloor} n^{(3)} \mu_{n-3}' + \frac{A_4}{\lfloor 4 \rfloor} n^{(4)} \mu_{n-4}' + \dots \\ + \frac{A_r}{\lfloor r \rfloor} n^{(r)} \mu_{n-r}'. \end{aligned}$$

Using the summation symbol, we have

$$(511) \quad \mu_n = \mu_n' + \sum_{i=3}^{n:r} \frac{A_i}{\lfloor i \rfloor} n^{(i)} \mu_{n-i}'$$

or

$$\mu_n = \mu_n' + \lfloor n \cdot \sum_{i=3}^{n:r} \frac{A_i}{i \lfloor n-i \rfloor} \mu_{n-i}' ;$$



which proves the lemma.

Since

$$\mu_2 = \mu_2',$$

the proof for equation (505) follows at once from the preceding lemma.

Having evaluated both the observed and the theoretical alphas, we proceed to compute the probable errors. We know that in general

(512) Probable error = .67449  $\times$  the standard deviation.

The standard deviation of the  $s$ th alpha is given by the equation

$$(513) \quad \sigma_{\alpha_s} = \frac{1}{\sqrt{N}} \left\{ \alpha_{2s} + s^2 \alpha_{s-1}^2 + s^2 \alpha_s \alpha_{s-1} + \frac{s^2}{4} \alpha_s^2 - \frac{(s-2)^2}{4} \alpha_s^2 - 2s \alpha_{s+1} \alpha_{s-1} - s \alpha_s \alpha_{s+2} \right\}^{1/2},$$

the proof of which is given below.

By definition

$$(514) \quad \alpha_s = \frac{\mu_s}{\sigma^s} = \frac{\mu_s}{\mu_2^{s/2}};$$

therefore

$$(515) \quad \delta_{\alpha_s} = \frac{\mu_2^{s/2} \delta \mu_s - \mu_s \cdot \frac{s}{2} \mu_2^{(s/2)-1} \delta \mu_2}{\mu_2^s} \\ = \frac{1}{\mu_2^{s/2}} \left\{ \delta \mu_s - \frac{s}{2} \cdot \frac{\mu_s}{\mu_2} \delta \mu_2 \right\}.$$

Square both sides of the equation, and obtain

$$(516) \quad (\delta \alpha_s)^2 = \frac{1}{\mu_2^s} \left\{ (\delta \mu_s)^2 - s \frac{\mu_s}{\mu_2} \delta \mu_s \delta \mu_2 + \frac{s^2 \mu_s^2}{4 \mu_2^2} (\delta \mu_2)^2 \right\};$$

that is,

$$(517) \quad \sigma_{\alpha_s}^2 = \frac{1}{\mu_2^s} \left\{ \sigma_{\mu_s}^2 - s \frac{\mu_s}{\mu_2} \sigma_{\mu_s} \sigma_{\mu_2} r_{\mu_s \mu_2} + \frac{s^2}{4} \cdot \frac{\mu_s^2}{\mu_2^2} \sigma_{\mu_2}^2 \right\}.$$

But we know that<sup>1</sup>

$$(518) \quad \sigma_{\mu_q} = \left\{ \frac{\mu_{2q} - \mu_q^2 - 2q\mu_{q+1}\mu_{q-1} + q^2\sigma^2 \cdot \mu_{q-1}^2}{N} \right\}^{1/2},$$

<sup>1</sup> On the Probable Errors of Frequency Constants. BIOMETRIKA, Editorial, volume II, 1902-03, page 276.

and that <sup>1</sup>

$$(519) \quad \sigma_{\mu_q} \sigma_{\mu_{q'}} r_{\mu_q \mu_{q'}} = \left\{ \frac{\mu_{q+q'} - \mu_q \mu_{q'} + qq' \sigma^2 \cdot \mu_{q-1} \mu_{q'-1}}{N} \right\}.$$

Whence

$$(520) \quad \begin{aligned} \sigma_{\mu_s}^2 &= \frac{1}{N} \{ \mu_{2s} - \mu_s^2 - 2s\mu_{s+1}\mu_{s-1} + s^2 \sigma^2 \cdot \mu_{q-1}^2 \}, \\ \sigma_{\mu_s} \sigma_{\mu_s} r_{\mu_s \mu_s} &= \frac{1}{N} \{ \mu_{s+2} - \sigma^2 \cdot \mu_s - s\mu_{s-1}\mu_{s+1} \}, \\ \sigma_{\mu_2}^2 &= \frac{1}{N} \{ \mu_4 - \sigma^4 \}. \end{aligned}$$

Substituting these values in (517) and simplifying, we obtain the following expression for the standard deviation of the  $s$ th theoretical alpha:

$$\sigma_{\alpha_s} = \frac{1}{\sqrt{N}} \left\{ \alpha_{2s} + s^2 \alpha_{s-1}^2 + s^2 \alpha_3 \alpha_{s-1} + \frac{s^2}{4} \alpha_4 \alpha_s^2 - \frac{(s-2)^2}{4} \alpha_s^2 - 2s\alpha_{s+1}\alpha_{s-1} - s\alpha_s \alpha_{s+2} \right\}^{1/2}.$$

We shall now apply our criterion to the graduation of the first numerical example, chapter one.

*First Step.* We have three theoretical distributions. Each one of these is governed by a definite mathematical law; each gives rise to a set of theoretical alphas which may be evaluated by means of equation (505). Since Charlier's Type A curve is the graduating function employed in this example, the formulas to be used in determining the theoretical alphas may be derived from (505) by merely substituting for  $\alpha_n'$  the alphas of the Laplacean probability function and giving to  $r$  the values 0, 3 and 4, successively. The resulting equations follow.

### I. First theoretical distribution

$$r = 0$$

$$(521) \quad \begin{array}{ll} = 0 & = 945 \\ = 3 & = 10395 \\ = 15 & = 135135 \\ = 105 & = 2027025 \\ & = 0 \end{array}$$

<sup>1</sup> *Ibid.* Page 277.

II. *Second theoretical distribution*

$$r = 3$$

$$\begin{array}{ll}
 \alpha_3 = \bar{\alpha}_3 & \alpha_{10} = 945 \\
 \alpha_4 = 3 & \alpha_{11} = 17325\bar{\alpha}_3 \\
 \alpha_5 = 10\bar{\alpha}_3 & \alpha_{12} = 10395 \\
 (522) \quad \alpha_6 = 15 & \alpha_{13} = 270270\bar{\alpha}_3 \\
 \alpha_7 = 105\bar{\alpha}_3 & \alpha_{14} = 135135 \\
 \alpha_8 = 105 & \alpha_{15} = 4729725\bar{\alpha}_3 \\
 \alpha_9 = 1260\bar{\alpha}_3 & \alpha_{16} = 2027025
 \end{array}$$

 III. *Third theoretical distribution*

$$r = 4$$

$$\begin{array}{ll}
 \alpha_3 = \bar{\alpha}_3 & \alpha_{10} = 3150\alpha_4 - 8505 \\
 \alpha_4 = \bar{\alpha}_4 & \alpha_{11} = 17325\bar{\alpha}_3 \\
 \alpha_5 = 10\bar{\alpha}_3 & \alpha_{12} = 51975\alpha_4 - 145530 \\
 (523) \quad \alpha_6 = 15\bar{\alpha}_4 - 30 & \alpha_{13} = 270270\bar{\alpha}_3 \\
 \alpha_7 = 105\bar{\alpha}_3 & \alpha_{14} = 945945\bar{\alpha}_4 - 2702700 \\
 \alpha_8 = 210\bar{\alpha}_4 - 525 & \alpha_{15} = 4729725\bar{\alpha}_3 \\
 \alpha_9 = 1260\bar{\alpha}_3 & \alpha_{16} = 18918900\bar{\alpha}_4 - 54729675
 \end{array}$$

The values of the theoretical alphas as computed by means of equations (521), (522) and (523) are given in Table 12. These are given up to and including  $\alpha_{16}$  although the first eight observed alphas only are known; for we need the value of  $\alpha_{28}$  in determining the probable error of  $\alpha_8$ , as may be readily seen from equation (513).

*Second Step.* With the first sixteen theoretical alphas computed, it is simply a matter of substitution and arithmetical calculation to find the standard deviations of the first eight theoretical alphas and then the corresponding probable errors. These are given in Table 13.

*Third Step.* Now we are ready to compare the absolute values of the variations with the probable errors. In Table 14 we have the variations and the probable errors placed in juxtaposition, thus making it easy to make a comparison between them.

TABLE 12

TABLE OF OBSERVED ALPHAS AND THEORETICAL ALPHAS

*First Numerical Example*

| $s$ | Observed<br>Alphas<br>$\alpha_s$ | Theoretical Alphas ( $\alpha_s$ ) of |                      |                     |
|-----|----------------------------------|--------------------------------------|----------------------|---------------------|
|     |                                  | First<br>Graduation                  | Second<br>Graduation | Third<br>Graduation |
| (1) | (2)                              | (3)                                  | (4)                  | (5)                 |
| 3   | .11                              | 0.00                                 | .11                  | .11                 |
| 4   | 3.20                             | 3.00                                 | 3.00                 | 3.20                |
| 5   | 1.32                             | 0.00                                 | 1.14                 | 1.14                |
| 6   | 18.02                            | 15.00                                | 15.00                | 17.95               |
| 7   | 14.83                            | 0.00                                 | 11.94                | 11.94               |
| 8   | 140.46                           | 105.00                               | 105.00               | 146.36              |
| 9   | —                                | 0.00                                 | 143.26               | 143.26              |
| 10  | —                                | 945.00                               | 945.00               | 1565.38             |
| 11  | —                                | 0.00                                 | 1969.87              | 1969.87             |
| 12  | —                                | 10395.00                             | 10395.00             | 20631.27            |
| 13  | —                                | 0.00                                 | 30730.02             | 30730.02            |
| 14  | —                                | 135135.00                            | 135135.00            | 321435.18           |
| 15  | —                                | 0.00                                 | 537775.27            | 537775.27           |
| 16  | —                                | 2027025.00                           | 2027025.00           | 5753028.6           |

TABLE 13

PROBABLE ERRORS OF THE ALPHAS

*First Numerical Example*

| $s$ | Probable Errors of $\alpha_s$ : $67449 \times \sigma_{\alpha_s}$ |                   |                  |
|-----|------------------------------------------------------------------|-------------------|------------------|
|     | First Graduation                                                 | Second Graduation | Third Graduation |
| (1) | (2)                                                              | (3)               | (4)              |
| 3   | .02059                                                           | .02026            | .02306           |
| 4   | .04117                                                           | .04099            | .05106           |
| 5   | .22551                                                           | .21966            | .29262           |
| 6   | .65747                                                           | .65946            | .95148           |
| 7   | 2.96075                                                          | 2.88061           | 4.55968          |
| 8   | 10.83855                                                         | 10.80878          | 18.44643         |

TABLE 14

TABLE SHOWING THE PROBABLE ERRORS AND THE DEVIATIONS OF THE ALPHAS

*First Numerical Example*

| <i>s</i> | Function            | First Graduation | Second Graduation | Third Graduation |
|----------|---------------------|------------------|-------------------|------------------|
| (1)      | (2)                 | (3)              | (4)               | (5)              |
| 3        | Probable Error..... | .02              | .02               | .02              |
|          | Variation.....      | .11              | .00               | .00              |
| 4        | Probable Error..... | .04              | .04               | .05              |
|          | Variation.....      | .20              | .20               | .00              |
| 5        | Probable Error..... | .23              | .22               | .29              |
|          | Variation.....      | 1.32             | .18               | .18              |
| 6        | Probable Error..... | .66              | .65               | .95              |
|          | Variation.....      | 3.02             | 3.02              | .07              |
| 7        | Probable Error..... | 2.96             | 2.88              | 4.56             |
|          | Variation.....      | 14.83            | 2.89              | 2.89             |
| 8        | Probable Error..... | 10.84            | 10.81             | 18.45            |
|          | Variation.....      | 35.46            | 35.46             | 5.90             |

From the preceding table we see readily that, if we take the first eight alphas of the three theoretical distributions into consideration, the third theoretical distribution gives the best fit. In the first graduation all the variations of the alphas exceed their respective probable errors. In the second the variations of  $\alpha_3$  and  $\alpha_5$  are smaller than their probable errors, but  $\alpha_4$ ,  $\alpha_6$ ,  $\alpha_7$  and  $\alpha_8$  have variations greater than the probable errors. In the third graduation the first eight theoretical alphas are identical to the observed alphas except for slight insignificant variations. We then conclude that the third theoretical distribution is the only reasonable graduation of the three.

It must be added that the theoretical alphas of the third theoretical distribution are not necessarily the best theoretical alphas which it is possible to obtain. We have only shown that the first eight alphas of the third graduation agree with the first eight alphas of the observed distribution, proper allowance being made, of course, to *reasonable variations*. In

other words, by using an expression of three terms as a graduating formula, we have obtained results just as good as what we would have obtained had we used an expression of seven terms instead.

TABLE 15

TABLE OF OBSERVED ALPHAS AND THEORETICAL ALPHAS

*Second Numerical Example, Chapter One.**Generating Function Employed:*  $\varphi(t) = \kappa e^{-(t^2/2)}$ .

| $s$ | Observed<br>Alphas<br>( $\bar{\alpha}_s$ ) | Theoretical Alphas ( $\alpha_s$ ) of   |                      |                     |                      |
|-----|--------------------------------------------|----------------------------------------|----------------------|---------------------|----------------------|
|     |                                            | First<br>Graduation<br>( $\alpha_s'$ ) | Second<br>Graduation | Third<br>Graduation | Fourth<br>Graduation |
| (1) | (2)                                        | (3)                                    | (4)                  | (5)                 | (6)                  |
| 3   | .21596                                     | 0.00000                                | .21596               | .21596              | .21596               |
| 4   | 3.00459                                    | 3.00000                                | 3.00000              | 3.00459             | 3.00459              |
| 5   | 2.25798                                    | 0.00000                                | 2.15961              | 2.15961             | 2.25798              |
| 6   | 15.31770                                   | 15.00000                               | 15.00000             | 15.06883            | 15.06883             |
| 7   | 23.59274                                   | 0.00000                                | 22.67594             | 22.67594            | 24.74163             |
| 8   | 111.59361                                  | 105.00000                              | 105.00000            | 105.96354           | 105.96354            |

TABLE 16

TABLE OF OBSERVED ALPHAS AND THEORETICAL ALPHAS

*Third Numerical Example, Chapter Two.**Generating Function Employed:*  $\varphi(x) = \kappa e^{-(x/\gamma)^{100/59}}$ .

| $s$ | Observed Alphas<br>( $\bar{\alpha}_s$ ) | Theoretical Alphas ( $\alpha_s$ ) of |                   |
|-----|-----------------------------------------|--------------------------------------|-------------------|
|     |                                         | First Graduation ( $\alpha_s'$ )     | Second Graduation |
| (1) | (2)                                     | (3)                                  | (4)               |
| 3   | .26784                                  | 0.00000                              | .26784            |
| 4   | 3.41558                                 | 3.38694                              | 3.38694           |
| 5   | 3.13180                                 | 0.00000                              | 2.67840           |
| 6   | 20.39578                                | 20.55225                             | 20.55225          |
| 7   | 34.89643                                | 0.00000                              | 31.75053          |
| 8   | 166.98504                               | 183.83160                            | 183.83160         |

In applying the above test to the results of the other numerical examples given in this paper, all that is necessary would be to change slightly the expressions for  $\alpha_n$ , substituting for  $\alpha_n'$  the alphas of the generating function employed instead of the Laplacean probability function. We shall not take the



time to give another practical application of our criterion, but it would be interesting to make a comparison between the observed alphas and the theoretical alphas in each of the other numerical examples given in the preceding pages. Tables 15, 16, 17 and 18 will be found useful for such an end.

TABLE 17

TABLE OF OBSERVED ALPHAS AND THEORETICAL ALPHAS

*Fourth Numerical Example, Chapter Four.*

*Generating Function Employed:*  $\varphi(x) = \kappa \operatorname{sech}^3 \gamma x$ .

| <i>s</i> | Observed<br>Alphas<br>( $\bar{\alpha}_s$ ) | Theoretical Alphas ( $\alpha_s$ ) of   |                      |                     |                      |
|----------|--------------------------------------------|----------------------------------------|----------------------|---------------------|----------------------|
|          |                                            | First<br>Graduation<br>( $\alpha_s'$ ) | Second<br>Graduation | Third<br>Graduation | Fourth<br>Graduation |
| (1)      | (2)                                        | (3)                                    | (4)                  | (5)                 | (6)                  |
| 3        | .58297                                     | 0.00000                                | .58297               | .58297              | .58297               |
| 4        | 3.69754                                    | 3.80625                                | 3.80625              | 3.69754             | 3.69754              |
| 5        | 6.79232                                    | 0.00000                                | 5.82970              | 5.82970             | 6.79232              |
| 6        | 29.80931                                   | 30.49496                               | 30.49496             | 28.86431            | 28.86431             |
| 7        | 92.43271                                   | 0.00000                                | 77.66253             | 77.66253            | 97.87755             |
| 8        | 387.50202                                  | 425.07204                              | 425.07204            | 386.10762           | 396.10762            |

TABLE 18

TABLE OF OBSERVED ALPHAS AND THEORETICAL ALPHAS

*Fifth Numerical Example, Chapter Four.*

*Generating Functions Employed:*  $\varphi(x) = \kappa \operatorname{sech}^4 \gamma x$ .

| <i>s</i> | Observed<br>Alphas<br>( $\bar{\alpha}_s$ ) | Theoretical Alphas ( $\alpha_s$ ) of   |                      |                     |                      |
|----------|--------------------------------------------|----------------------------------------|----------------------|---------------------|----------------------|
|          |                                            | First<br>Graduation<br>( $\alpha_s'$ ) | Second<br>Graduation | Third<br>Graduation | Fourth<br>Graduation |
| (1)      | (2)                                        | (3)                                    | (4)                  | (5)                 | (6)                  |
| 3        | .58297                                     | 0.00000                                | .58297               | .58297              | .58297               |
| 4        | 3.69754                                    | 3.59376                                | 3.59376              | 3.69754             | 3.69754              |
| 5        | 6.79232                                    | 0.00000                                | 5.82970              | 5.82970             | 6.79232              |
| 6        | 29.80931                                   | 25.84599                               | 25.84599             | 27.40265            | 27.40265             |
| 7        | 92.46271                                   | 0.00000                                | 73.32690             | 73.32690            | 93.54185             |
| 8        | 387.50202                                  | 311.18468                              | 311.18468            | 337.29119           | 337.29119            |

The next question is concerned with the limiting form of the generalized-hyperbolic-secant curve as we make the exponent of the hyperbolic secant approach infinity as a limit.

It will be recalled that the general expressions for  ${}_{2n}\alpha_4$  and  ${}_{2n+1}\alpha_4$  of the generalized-hyperbolic-secant are

$$(524) \quad \begin{aligned} {}_{2n}\alpha_4 &= 3 \left[ 1 + \frac{\frac{\pi^4}{90} - \sum_{s=1}^{n-1} \frac{1}{s^4}}{\left\{ \frac{\pi^2}{6} - \sum_{s=1}^{n-1} \frac{1}{s^2} \right\}^2} \right]; \\ {}_{2n+1}\alpha_4 &= 3 \left[ 1 + \frac{\frac{\pi^4}{96} - \sum_{s=0}^{n-1} \frac{1}{(2s+1)^4}}{\left\{ \frac{\pi^2}{8} - \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2} \right\}^2} \right]; \end{aligned}$$

where

$${}_{2n}\alpha_4 \equiv \frac{{}_{2n}\mu_4}{\{{}_{2n}\mu_2\}^2},$$

and  ${}_{2n+1}\alpha_4$  stands for the corresponding function for the odd powers of the hyperbolic secant.

The above expressions are derived in the following manner. By definition

$${}_{2n}\alpha_4 = \frac{{}_{2n}\mu_4}{\{{}_{2n}\mu_2\}^2};$$

whence, substituting the respective values for the second and the fourth moments found in chapter four—see equation (414)—we obtain the equation

$$(525) \quad {}_{2n}\alpha_4 = \frac{\frac{7\pi^4}{240\gamma^4} - \frac{\pi^2}{4\gamma^4} \cdot {}_nS_2 + \frac{3}{4\gamma^4} ({}_nS_2)^2 - \frac{3}{4\gamma^4} \cdot S_4}{\left\{ \frac{\pi^2}{12\gamma^2} - \frac{1}{2\gamma^2} \cdot {}_nS_2 \right\}^2},$$

where

$${}_nS_r \equiv \frac{1}{1^r} + \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{(n-1)^r}.$$

Simplifying, we get

$${}_{2n}\alpha_4 = \frac{\frac{7\pi^4}{60} - \pi^2 \cdot {}_nS_2 + 3({}_nS_2)^2 - 3 \cdot {}_nS_4}{\left\{ \frac{\pi^2}{6} - {}_nS_2 \right\}^2}$$

$$\begin{aligned}
 (526) \quad &= 3 \left[ \frac{\frac{7}{180} \pi^4 - \frac{\pi^2}{3} \cdot {}_n S_2 + ({}_n S_2)^2 - {}_n S_4}{\left\{ \frac{\pi^2}{6} - {}_n S_2 \right\}^2} \right] \\
 &= 3 \left[ \frac{\frac{\pi^4}{36} - \frac{\pi^2}{3} \cdot {}_n S_2 + ({}_n S_2)^2 + \frac{\pi^4}{90} - {}_n S_4}{\left\{ \frac{\pi^2}{6} - {}_n S_2 \right\}^2} \right] \\
 &= 3 \left[ 1 + \frac{\frac{\pi^4}{90} - {}_n S_4}{\left\{ \frac{\pi^2}{6} - {}_n S_2 \right\}^2} \right].
 \end{aligned}$$

In a similar manner the following expression for  ${}_{2n+1}\alpha_4$  may be deduced:

$$\begin{aligned}
 (527) \quad {}_{2n+1}\alpha_4 &= \frac{\frac{5\pi^4}{16\gamma^4} - \frac{3\pi^2}{\gamma^4} \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2}}{\frac{1}{\gamma^4} \sum_{s=1}^{n-1} \frac{1}{(2s+1)^2} \sum_{r=0}^{s-1} \frac{1}{(2r+1)^2}} \\
 &\quad + \frac{\frac{\pi^2}{4\gamma^2} - \frac{2}{\gamma^2} \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2}}{\left\{ \frac{\pi^2}{4\gamma^2} - \frac{2}{\gamma^2} \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2} \right\}^2} \\
 &= \frac{5\pi^4 - 48\pi^2 \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2}}{+ 192 \left( \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2} \right)^2 - 192 \sum_{s=0}^{n-1} \frac{1}{(2s+1)^4}} \\
 &\quad + \frac{\pi^2 - 8 \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2}}{\left\{ \pi^2 - 8 \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2} \right\}^2} \\
 &= 3 \left[ 1 + \frac{\frac{\pi^4}{96} - \sum_{s=0}^{n-1} \frac{1}{(2s+1)^4}}{\left\{ \frac{\pi^2}{8} - \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2} \right\}^2} \right].
 \end{aligned}$$

Or, expressing (526) and (527) in slightly different forms, we have

$$\begin{aligned}
 (528) \quad {}_{2n}\alpha_4 &= 3 + 3 \left[ \frac{\frac{\pi^4}{90} - {}_nS_4}{\left\{ \frac{\pi^2}{6} - {}_nS_2 \right\}^2} \right]; \\
 {}_{2n+1}\alpha_4 &= 3 + 3 \left[ \frac{\frac{\pi^4}{96} - \sum_{s=0}^{n-1} \frac{1}{(2s+1)^4}}{\left\{ \frac{\pi^2}{8} - \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2} \right\}^2} \right].
 \end{aligned}$$

We shall take the expression for  ${}_{2n}\alpha_4$  and examine the second term more closely,

$$(529) \quad 3 \left[ \frac{\frac{\pi^4}{90} - {}_nS_4}{\left\{ \frac{\pi^2}{6} - {}_nS_2 \right\}^2} \right].$$

We know that <sup>1</sup>

$$(530) \quad \lim_{n \rightarrow \infty} {}_nS_4 = \frac{\pi^4}{90}; \quad \text{and} \quad \lim_{n \rightarrow \infty} {}_nS_2 = \frac{\pi^2}{6}.$$

Now build up the double series

$$\begin{aligned}
 (531) \quad & \frac{1}{1^2 \cdot 1^2} + \frac{1}{1^2 \cdot 2^2} + \frac{1}{1^2 \cdot 3^2} + \frac{1}{1^2 \cdot 4^2} + \cdots + \frac{1}{1^2 \cdot n^2} + \cdots \\
 & \frac{1}{2^2 \cdot 1^2} + \frac{1}{2^2 \cdot 2^2} + \frac{1}{2^2 \cdot 3^2} + \frac{1}{2^2 \cdot 4^2} + \cdots + \frac{1}{2^2 \cdot n^2} + \cdots \\
 & \frac{1}{3^2 \cdot 1^2} + \frac{1}{3^2 \cdot 2^2} + \frac{1}{3^2 \cdot 3^2} + \frac{1}{3^2 \cdot 4^2} + \cdots + \frac{1}{3^2 \cdot n^2} + \cdots \\
 & \frac{1}{4^2 \cdot 1^2} + \frac{1}{4^2 \cdot 2^2} + \frac{1}{4^2 \cdot 3^2} + \frac{1}{4^2 \cdot 4^2} + \cdots + \frac{1}{4^2 \cdot n^2} + \cdots \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & \frac{1}{n^2 \cdot 1^2} + \frac{1}{n^2 \cdot 2^2} + \frac{1}{n^2 \cdot 3^2} + \frac{1}{n^2 \cdot 4^2} + \cdots + \frac{1}{n^2 \cdot n^2} + \cdots \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
 & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
 \end{aligned}$$

<sup>1</sup> See Chrystal's *Algebra*, vol. II, page 367. London, 1906.

This series is absolutely convergent. The diagonal forms a series

$$(532) \quad \frac{1}{1^2 \cdot 1^2} + \frac{1}{2^2 \cdot 2^2} + \frac{1}{3^2 \cdot 3^2} + \cdots + \frac{1}{n^2 \cdot n^2} + \cdots,$$

the limiting value of which, when the number of terms approaches infinity, is  $\pi^4/90$ . The expression for  ${}_{2n}\alpha_4$  may now be expressed verbally as the sum of two quantities; the first is the constant number 3, and the second is a variable quantity and is equal to three times the ratio of the residue of the diagonal series (532) to the residue of the entire double series (531). This second quantity approaches zero as a limit when  $n$  is increased indefinitely. It follows that

$$(533) \quad \lim_{n \rightarrow \infty} {}_{2n}\alpha_4 = 3.$$

In a similar manner, it may be shown that

$$(534) \quad \lim_{n \rightarrow \infty} {}_{2n+1}\alpha_4 = 3.$$

In other words, as  $m$  approaches infinity, the value of  ${}_m\alpha_4$  for the generalized-hyperbolic-secant curve approaches as a limit the value of  $\alpha_4$  for Charlier's Type A curve. Furthermore it can be proved that  ${}_m\alpha_6, {}_m\alpha_8, \cdots, {}_m\alpha_{2n}$  of the generalized-hyperbolic-secant approach  $\alpha_6, \alpha_8, \cdots, \alpha_{2n}$ , respectively, of the normal probability function, as  $m$  increases infinitely in value, provided  $n$  remains finite. This, together with the fact that there are certain geometrical characteristics common to both the generalized-hyperbolic-secant and the normal probability function, impels us to raise the question: Is the limiting form of  $\kappa \operatorname{sech}^m \gamma x$  when  $m$  approaches infinity the well-known Laplacean probability function? If not, what is it? And what relation does it have to the Laplacean probability function?

It has been suggested <sup>1</sup> that we utilize the relation

$$(535) \quad D^n = \left( \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \cdots \right)^n$$

<sup>1</sup> See Carver's *The Mathematical Representation of Frequency Distributions*, loc. cit., page 728.

and transform the series of derivatives into a series of differences. If such a transformation could be proved to be mathematically sound and legitimate, it would be of great practical value in statistical work. The  $n$ th difference of a function is in general easier to compute arithmetically than the  $n$ th derivative. But here are some of the questions which must be answered before we can proceed to make use of the transformed series: When the derivative-series is convergent, would the corresponding difference-series be convergent also? If a distribution is not representable by the derivative-series, can it not be represented by the corresponding difference-series? When both series are convergent, will they have the same rapidity of convergence, or will the transformation have an effect on the rapidity of convergence? An affirmative answer to the first question alone would be valuable. To answer all three of the questions is tantamount to laying down a set of necessary and sufficient conditions for a derivative-series to be transformed into a difference-series.

We shall close this chapter by evaluating the probability integral by means of the generalized-hyperbolic-secant curve,

$$(536) \quad y = \varphi(x) - \frac{a_3}{[3]} \varphi^3(x) + \frac{a_4}{[4]} \varphi^4(x) - \dots,$$

where

$$\varphi(x) = \kappa_m \operatorname{sech}^m \gamma_m x.$$

Since the probability function is even, all the odd coefficients  $a_{2n+1}$  disappear, and equation (536) becomes

$$(537) \quad y = \varphi(x) + \frac{a_4}{[4]} \varphi^4(x) + \frac{a_6}{[6]} \varphi^6(x).$$

The values of the argument chosen are 1, 2, 3 and 4. The final results for  $m$  equal to 1, 2, 5 and 10, respectively, are given in Table 19. The true values of the integral as given in Pearson's Tables for Statisticians and Biometricians are also given so that a comparison between these and the ones we have found can be easily made.

The values of the constants and the coefficients used are given below.



*Constants:*

| $m$ | $\kappa/\gamma$ | $\gamma$  | $\kappa$ | $\kappa\gamma^3$ | $\kappa\gamma^5$ |
|-----|-----------------|-----------|----------|------------------|------------------|
| 1   | .3183099        | 1.5707963 | .5000000 | 1.9378924        | 4.7815578        |
| 2   | .5000000        | .9068997  | .4534498 | .3382260         | .2781797         |
| 5   | .8488264        | .4951554  | .4203010 | .05102523        | .01251031        |
| 10  | 1.2304688       | .3326582  | .4093255 | .01506828        | .001667478       |

*Coefficients:*

| $m$ | $a_4/ 4$    | $a_6/ 6$     |
|-----|-------------|--------------|
| 1   | -.08333333  | -.02222222   |
| 2   | -.05000000  | -.00952381   |
| 5   | -.01939936  | -.001705296  |
| 10  | -.001911345 | -.0003927875 |

*Integrals:*

$$\begin{aligned}
 \frac{\kappa}{\gamma} \int_{-\infty}^t \operatorname{sech} t dt &= A(t), \quad t \equiv \gamma x; \\
 \frac{\kappa}{\gamma} \int_{-\infty}^t \operatorname{sech}^2 t dt &= \frac{\kappa}{\gamma} \{1 + \tanh t\}; \\
 (538) \quad \frac{\kappa}{\gamma} \int_{-\infty}^t \operatorname{sech}^5 t dt &= \frac{1}{8} \left( \frac{\kappa}{\gamma} \right) \tanh t \{2 \operatorname{sech}^3 t + 3 \operatorname{sech} t\} \\
 &\quad + A(t); \\
 \frac{\kappa}{\gamma} \int_{-\infty}^t \operatorname{sech}^{10} t dt &= \frac{\kappa}{\gamma} \left\{ \frac{128}{315} + \tanh t - \frac{4}{3} \tanh^3 t \right. \\
 &\quad \left. + \frac{6}{5} \tanh^5 t - \frac{4}{7} \tanh^7 t + \frac{1}{9} \tanh^9 t \right\}.
 \end{aligned}$$

The values of  $A(t)$  are taken from Table V, Appendix C. The values of the hyperbolic functions are derived from Pernot and Woods' Logarithms of hyperbolic functions to twelve significant figures and from Gudermann's tables published in Crelle's JOURNAL, volumes 8 and 9. Newton's interpolation formula is employed.

TABLE 19

COMPUTATION OF THE PROBABILITY INTEGRAL BY MEANS OF THE  
GENERALIZED-HYPERBOLIC-SECANT CURVE

| $x$ | True Value<br>$\int_{-\infty}^x \kappa e^{-(x^2/2)} dx$ | Approximate Value by Using an Expansion of |           |             |
|-----|---------------------------------------------------------|--------------------------------------------|-----------|-------------|
|     |                                                         | One Term                                   | Two Terms | Three Terms |
| (1) | (2)                                                     | (3)                                        | (4)       | (5)         |

  

|                                                           |         |         |         |         |
|-----------------------------------------------------------|---------|---------|---------|---------|
| A. $\varphi(x) = \kappa_1 \operatorname{sech} \gamma_1 x$ |         |         |         |         |
| 1                                                         | .841345 | .869518 | .872293 | .658580 |
| 2                                                         | .977250 | .972506 | .985766 | .990881 |
| 3                                                         | .998650 | .994281 | .997176 | .999048 |
| 4                                                         | .999968 | .998810 | .999413 | .999810 |

  

|                                                                   |         |         |         |         |
|-------------------------------------------------------------------|---------|---------|---------|---------|
| B. $\varphi(x) = \kappa_2 \operatorname{sech}^2 \gamma_2 \cdot x$ |         |         |         |         |
| 1                                                                 | .841345 | .859821 | .838870 | .809399 |
| 2                                                                 | .977250 | .974108 | .983133 | .980828 |
| 3                                                                 | .998650 | .995685 | .997871 | .998953 |
| 4                                                                 | .999968 | .999294 | .999672 | .999901 |

  

|                                                                   |         |         |         |         |
|-------------------------------------------------------------------|---------|---------|---------|---------|
| C. $\varphi(x) = \kappa_5 \operatorname{sech}^5 \gamma_5 \cdot x$ |         |         |         |         |
| 1                                                                 | .841345 | .849484 | .839195 | .840738 |
| 2                                                                 | .977250 | .975714 | .978870 | .977430 |
| 3                                                                 | .998650 | .997294 | .998433 | .998651 |
| 4                                                                 | .999968 | .999746 | .999898 | .999960 |

  

|                                                                            |         |         |         |         |
|----------------------------------------------------------------------------|---------|---------|---------|---------|
| D. $\varphi(x) = \kappa_{10} \operatorname{sech}^{10} \gamma_{10} \cdot x$ |         |         |         |         |
| 1                                                                          | .841345 | .845433 | .840722 | .841259 |
| 2                                                                          | .977250 | .976421 | .977705 | .977322 |
| 3                                                                          | .998650 | .997967 | .998582 | .998628 |
| 4                                                                          | .999968 | .999880 | .999950 | .999968 |

It is to be noted that the greater the value of  $m$  taken the closer the values found to the true values of the integral. This is to be expected since the  $\alpha_4$  of the probability function is 3, and since the  $m\alpha_4$  of the generalized-hyperbolic-secant becomes close to 3 as  $m$  increases in value. When  $m$  is 10,  $m\alpha_4$  has a value of 3.21872. Here then we have a generating function the  $\alpha_4$  of which is .21872 greater than the corresponding alpha of the function to be evaluated and we have, by using five terms of the series-expansion, obtained values of the function correct to at least four significant figures.

When the argument has a value of 4, the computed value of the function agrees with the true value to six significant figures.

The question now arises: To what extent can we use statistical generating functions in pure mathematics? We have just shown the application of the generalized-hyperbolic-secant curve to the evaluation of a definite integral when the indefinite integral can not be found. It is true that in the example given we have avoided a number of questions by choosing to evaluate a function the true values of which are already tabulated by other investigators. Ordinarily, questions such as those connected with uniform convergence will have to be investigated before we can proceed to evaluate the function by means of the series-expansion. We have evaluated the probability integral, and all we have done to justify our results is to compare them with the values found by other writers. In this we have merely endeavored to point out what we think is worth further study: the application of statistical generating functions to problems in pure mathematics.

## APPENDIX A

The following extract from Carslaw's treatise on Fourier's series and integrals gives in a concise manner the conditions for an arbitrary function to be expressible as a Fourier's double integral:

"If the interval  $-\infty < x < \infty$  can be broken up into the intervals

$$-\infty < x < -a', \quad -a' < x < a, \quad a < x < \infty,$$

in which  $a'$ ,  $a$  are finite and positive, such that in the first and third intervals  $f(x)$  is monoclinic and always remains of the same sign and the integrals

$$\int_{-\infty}^{-a'} f(x') dx' \quad \text{and} \quad \int_a^{\infty} f(x') dx'$$

are convergent, and if  $f(x)$  is finite and otherwise satisfies Dirichlet's Conditions in any finite interval, then

$$(A01) \quad \frac{1}{\pi} \int_0^{\infty} d\alpha \int_{-\infty}^{\infty} f(x') \cos \alpha(x' - x) dx' \\ = \frac{1}{2} [f(x - 0) + f(x + 0)],$$

where  $x$  may have any real value.

"If we put  $f(x) = 0$  in the intervals  $-\infty < x < -l'$  and  $l < x < \infty$ , it follows that when the arbitrary function is given in the interval  $-l' < x < l$ ,

$$(A02) \quad \frac{1}{\pi} \int_0^{\infty} d\alpha \int_{-l'}^l f(x') \cos \alpha(x' - x) dx' \\ = \frac{1}{2} [f(x - 0) + f(x + 0)],$$

when  $-l' < x < l$ ; and that when  $x = l$ , the value of the integral is  $\frac{1}{2}f(l - 0)$ , when  $x = -l'$ , the value of the integral is  $\frac{1}{2}f(-l' + 0)$ .

"In the same way we have

$$(A03) \quad \frac{1}{\pi} \int_0^\infty d\alpha \int_0^l f(x') \cos \alpha(x' - x) dx' \\ = \frac{1}{2} [f(x - 0) + f(x + 0)]$$

when  $0 < x < l$ , and at  $x = 0$  and  $x = l$  its value is  $\frac{1}{2}f(+0)$  and  $\frac{1}{2}f(l - 0)$ ."

The Dirichlet's Conditions referred to in the preceding paragraphs require that <sup>1</sup>

"(1) The function is continuous in its domain at every point, with the exception of a finite number of points at which it may have ordinary discontinuities,

"(2) The domain may be divided into a finite number of parts, such that in any one of them the function is monotone; or in accordance with the more usual but less exact expression, the function has only a finite number of maxima and minima in its domain."

It must be stated here that these conditions are sufficient but not necessary, and it is entirely possible that a frequency curve may exhibit certain geometrical properties that would make our second assumption in chapter one look unreasonable and still be expressible as a Fourier's double integral.

<sup>1</sup> *The Theory of Functions of a Real Variable*, Hobson, E. W. 1907. Page 664.

## APPENDIX B

In his article, Charlier has given the determining equations for  $S_r(x)$  for  $r = 1, 2, 3$ , and 4 only. We have carried his work further and have found the following equations for  $S_5(x)$  and  $S_6(x)$  respectively:

The following scheme is used in order to be consistent with the one used by Charlier. Charlier's evaluations are found right after equation (11) of his article.

$$r = 5$$

$$\begin{aligned}
 \Delta_5 &= \lambda_0^6, \\
 \lambda_0^6 \beta_0^{(5)} &= -\lambda_0^5, \\
 \lambda_0^6 \beta_1^{(5)} &= \lambda_1 \lambda_0^4, \\
 \lambda_0^6 \beta_2^{(5)} &= -\lambda_0^3 \lambda_1^2 + \lambda_0^4 \lambda_2, \\
 (A04) \quad \lambda_0^6 \beta_3^{(5)} &= \lambda_0^2 \lambda_1^3 - 2\lambda_0^3 \lambda_1 \lambda_2 + \lambda_0^3 \lambda_3, \\
 \lambda_0^6 \beta_4^{(5)} &= -\lambda_0 \lambda_1^4 + 3\lambda_0^2 \lambda_1^2 \lambda_2 - 2\lambda_0^3 \lambda_1 \lambda_3 \\
 &\quad - \lambda_0^3 \lambda_2^2 + \lambda_0^4 \lambda_4, \\
 \lambda_0^6 \beta_5^{(5)} &= \lambda_1^5 - 4\lambda_0 \lambda_1^3 \lambda_2 + 3\lambda_0^2 \lambda_1^2 \lambda_3 + 3\lambda_0^2 \lambda_1 \lambda_2^2 \\
 &\quad - 2\lambda_0^3 \lambda_1 \lambda_4 - 2\lambda_0^3 \lambda_2 \lambda_3 + \lambda_0^4 \lambda_5; \\
 \therefore \lambda_0^6 \cdot S_5(x) &= -\lambda_0^5 \frac{x^5}{[5]} + \lambda_1 \lambda_0^4 \frac{x^4}{[4]} - (\lambda_0^3 \lambda_1^2 - \lambda_0^4 \lambda_2) \frac{x^3}{[3]} \\
 &\quad + (\lambda_0^2 \lambda_1^3 - 2\lambda_0^3 \lambda_1 \lambda_2 + \lambda_0^3 \lambda_3) \frac{x^2}{[2]} \\
 (A05) \quad &\quad - (\lambda_0 \lambda_1^4 - 3\lambda_0^2 \lambda_1^2 \lambda_2 + 2\lambda_0^3 \lambda_1 \lambda_3 \\
 &\quad + \lambda_0^3 \lambda_2^2 - \lambda_0^4 \lambda_4) \frac{x}{[1]} \\
 &\quad + \lambda_1^5 - 4\lambda_0 \lambda_1^3 \lambda_2 + 3\lambda_0^2 \lambda_1^2 \lambda_3 + 3\lambda_0^2 \lambda_1 \lambda_2^2 \\
 &\quad - 2\lambda_0^3 \lambda_1 \lambda_4 - 2\lambda_0^3 \lambda_2 \lambda_3 + \lambda_0^4 \lambda_5.
 \end{aligned}$$

$$r = 6$$

$$\begin{aligned}
 \Delta_6 &= \lambda_0^7, \\
 \lambda_0^7 \beta_0^{(6)} &= \lambda_0^6, \\
 \lambda_0^7 \beta_1^{(6)} &= -\lambda_1 \lambda_0^5, \\
 \lambda_0^7 \beta_2^{(6)} &= \lambda_0^4 \lambda_1^2 - \lambda_0^5 \lambda_2,
 \end{aligned}$$

$$\begin{aligned}
(406) \quad & \lambda_0^7 \beta_3^{(6)} = -\lambda_0^3 \lambda_1^3 + 2\lambda_0^4 \lambda_1 \lambda_2 - \lambda_0^4 \lambda_3, \\
& \lambda_0^7 \beta_4^{(6)} = \lambda_0^2 \lambda_1^4 - 3\lambda_0^3 \lambda_1^2 \lambda_2 + 2\lambda_0^4 \lambda_1 \lambda_3 \\
& \quad + \lambda_0^3 \lambda_2^2 - \lambda_0^4 \lambda_4, \\
& \lambda_0^7 \beta_5^{(6)} = -\lambda_0 \lambda_1^5 + 4\lambda_0^2 \lambda_1^3 \lambda_2 - 3\lambda_0^3 \lambda_1^2 \lambda_3 \\
& \quad - 3\lambda_0^3 \lambda_1 \lambda_2^2 + 2\lambda_0^4 \lambda_1 \lambda_4 + 2\lambda_0^4 \lambda_2 \lambda_3 - \lambda_0^5 \lambda_5, \\
& \lambda_0^7 \beta_6^{(6)} = \lambda_1^6 - 5\lambda_0 \lambda_1^4 \lambda_2 + 4\lambda_0^2 \lambda_1^3 \lambda_3 + 6\lambda_0^2 \lambda_1^2 \lambda_2^2 \\
& \quad - 3\lambda_0^3 \lambda_1^2 \lambda_4 - 6\lambda_0^3 \lambda_1 \lambda_2 \lambda_3 - \lambda_0^3 \lambda_2^3 \\
& \quad + 2\lambda_0^4 \lambda_1 \lambda_5 + 2\lambda_0^4 \lambda_2 \lambda_4 + \lambda_0^4 \lambda_3^2 - \lambda_0^5 \lambda_6; \\
\therefore \quad & \lambda_0^7 S_6(x) = \lambda_0^6 \frac{x^6}{[6]} - \lambda_1 \lambda_0^5 \frac{x^5}{[5]} + (\lambda_0^4 \lambda_1^2 - \lambda_0^5 \lambda_2) \frac{x^4}{[4]} \\
& \quad - (\lambda_0^3 \lambda_1^3 - 2\lambda_0^4 \lambda_1 \lambda_2 + \lambda_0^4 \lambda_3) \frac{x^3}{[3]} \\
& \quad + (\lambda_0^2 \lambda_1^4 - 3\lambda_0^3 \lambda_1^2 \lambda_2 + 2\lambda_0^4 \lambda_1 \lambda_3 \\
& \quad \quad + \lambda_0^3 \lambda_2^2 - \lambda_0^4 \lambda_4) \frac{x^2}{[2]} \\
(407) \quad & \quad - (\lambda_0 \lambda_1^5 - 4\lambda_0^2 \lambda_1^3 \lambda_2 + 3\lambda_0^3 \lambda_1^2 \lambda_3 + 3\lambda_0^3 \lambda_1 \lambda_2^2 \\
& \quad - 2\lambda_0^4 \lambda_1 \lambda_4 - 2\lambda_0^4 \lambda_2 \lambda_3 + \lambda_0^5 \lambda_5) \frac{x}{[1]} \\
& \quad + \lambda_1^6 - 5\lambda_0 \lambda_1^4 \lambda_2 + 4\lambda_0^2 \lambda_1^3 \lambda_3 + 6\lambda_0^2 \lambda_1^2 \lambda_2^2 \\
& \quad - 3\lambda_0^3 \lambda_1^2 \lambda_4 - 6\lambda_0^3 \lambda_1 \lambda_2 \lambda_3 - \lambda_0^3 \lambda_2^3 \\
& \quad + 2\lambda_0^4 \lambda_1 \lambda_5 + 2\lambda_0^4 \lambda_2 \lambda_4 + \lambda_0^4 \lambda_3^2 - \lambda_0^5 \lambda_6.
\end{aligned}$$

When the auxiliary function  $f(x)$  is even,

$$(408) \quad \lambda_{2n+1} = \frac{1}{[2n+1]} \int_{-\infty}^{+\infty} x^{2n+1} f(x) dx$$

disappears, and the determining equations for  $S_5(x)$  and  $S_6(x)$  then take the following forms:

$$(409) \quad \lambda_0^6 \cdot S_5(x) = -\lambda_0^5 \frac{x^5}{[5]} + \lambda_0^4 \lambda_2 \frac{x^3}{[3]} - (\lambda_0^3 \lambda_2^2 - \lambda_0^4 \lambda_4) \frac{x}{[1]},$$

$$\begin{aligned}
(410) \quad \lambda_0^7 \cdot S_6(x) = & \lambda_0^6 \frac{x^6}{[6]} - \lambda_0^5 \lambda_2 \frac{x^4}{[4]} + (\lambda_0^3 \lambda_2^2 - \lambda_0^4 \lambda_4) \frac{x^2}{[2]} \\
& - (\lambda_0^3 \lambda_2^3 - 2\lambda_0^4 \lambda_2 \lambda_4 + \lambda_0^5 \lambda_6).
\end{aligned}$$



## APPENDIX C

The mathematical tables given in the following pages will be found useful in graduating frequency distributions by the methods suggested in this paper.

Table I. Reference Table for the Generalized Curve of Error.

Table II. Reference Table for the Generalized-Hyperbolic-Secant Curve.

The following formulas were used in constructing the different columns:

(a) When  $m$  is even;  $m \leq 4$ .

$$(A11) \quad {}_{2n}\alpha_4 = 3 \left[ 1 + \frac{\frac{\pi^4}{90} - {}_nS_4}{\left\{ \frac{\pi^2}{6} - {}_nS_2 \right\}^2} \right], \quad 2n \equiv m;$$

$$(A12) \quad \begin{aligned} \gamma \cdot \sigma &= \left[ \frac{\pi^2}{12} - \frac{1}{2} \cdot {}_nS_2 \right]^{1/2}; \\ \frac{\kappa}{\gamma} &= \frac{1}{2^{2n-1}} \cdot \frac{2n-1}{(n-1)^2}; \\ {}_nS_r &= \frac{1}{1^r} + \frac{1}{2^r} + \frac{1}{3^r} + \cdots + \frac{1}{(n-1)^r}. \end{aligned}$$

(b) When  $m$  is odd;  $m \leq 3$ .

$$(A13) \quad {}_{2n+1}\alpha_4 = 3 \left[ 1 + \frac{\frac{\pi^4}{96} - \sum_{s=0}^{n-1} \frac{1}{(2s+1)^4}}{\left\{ \frac{\pi^2}{8} - \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2} \right\}^2} \right],$$

$(2n+1) \equiv m;$

$$(A14) \quad \begin{aligned} \gamma \sigma &= \left[ \frac{\pi^2}{4} - 2 \sum_{s=0}^{n-1} \frac{1}{(2s+1)^2} \right]^{1/2}; \\ \frac{\kappa}{\gamma} &= 2^{2n} \cdot \frac{(|n|)^2}{|2n|} \cdot \left( \frac{1}{\pi} \right). \end{aligned}$$

The derivation of (A11) and (A13) has been given in chapter five. The other formulas may be derived in the following manner:

By hypothesis

$${}_{2n}\mu_2 = \sigma^2;$$

that is,

$$(A15) \quad \frac{\pi^2}{12\gamma^2} - \frac{1}{2\gamma^2} \cdot {}_nS_2 = \sigma^2.$$

Solving, we have

$$\gamma\sigma = \left\{ \frac{\pi^2}{12} - \frac{1}{2} \cdot {}_nS_2 \right\}^{1/2}.$$

Again, by hypothesis

$$\int_{-\infty}^{\infty} \kappa \operatorname{sech}^{2n} \gamma x dx = 1.$$

Accordingly

$$(A16) \quad \begin{aligned} \kappa &= \frac{1}{{}_{2n}I_0} \\ &= \frac{(2n-1)(2n-3)(2n-5) \cdots 3}{(2n-2)(2n-4)(2n-6) \cdots 4} \cdot \frac{1}{{}_2I_0} \\ &= \frac{1}{2^{2n-2}} \left( \frac{2n-1}{(n-1)^2} \cdot \frac{1}{{}_2I_0} \right). \end{aligned}$$

But

$$(A17) \quad {}_2I_0 = \frac{2}{\gamma};$$

therefore

$$\kappa = \frac{1}{2^{2n-1}} \left( \frac{2n-1}{(n-1)^2} \right).$$

The corresponding formulas used when  $m$  is odd may be proven in a similar manner.

$$\text{Table III. } \sum_{s=1}^n \frac{1}{s^r}; \quad r = 2, 4, 6.$$

$$\text{Table IV. } \sum_{s=0}^n \frac{1}{(2s+1)^r}; \quad r = 2, 4, 6.$$

Table V. Area of the hyperbolic-secant curve in terms of the abscissa.

This table gives the area for negative values of the argument: To find the area when the abscissa is positive, use the equation

$$(A18) \quad A_{(t)} = 1 - A_{(-t)}.$$

The following formulas were used in constructing the table:

(a) When  $0 \succ |t| < .80$ :

$$(A19) \quad \begin{aligned} \frac{\pi}{2} \cdot A_{(-t)} = & + .7853982 & - .5t \\ & + .08333333t^3 & - .02083333t^5 \\ & + .0060515873t^7 & - .001908344t^9 \\ & + .0006328288t^{11} & - .0002170191t^{13} \\ & + .00007622730t^{15} & - .00002725920t^{17}. \end{aligned}$$

(b) When  $|t| \preccurlyeq .80$ :

$$(A20) \quad \frac{\pi}{2} \cdot A_{(-t)} = e^{-t} - \frac{1}{3}e^{-3t} + \frac{1}{5}e^{-5t} - \frac{1}{7}e^{-7t} + \dots$$

The first formula is obtained by MacLaurin's theorem, while the second is taken from Peirce's Table of Integrals, No. 779.

The values of the powers of the natural numbers used in connection with the construction of Tables III and IV are from Pearson's Tables for Statisticians and Biometricians, while those of  $e^{-rt}$  used in constructing Table V are from Smithsonian Mathematical Tables—Hyperbolic Functions, Second Reprint.

TABLE I

REFERENCE TABLE FOR GENERALIZED CURVE OF ERROR

| $p$ | $\alpha_4$ | $\log \Gamma p$ | $\log \Gamma_2 p$ | $\log \Gamma_3 p$ | $\log \Gamma_4 p$ |
|-----|------------|-----------------|-------------------|-------------------|-------------------|
| .30 | 2.321616   | .4758990        | 0.0288268         | $\bar{1}.9475449$ | 0.0197334         |
| .31 | 2.349981   | .4609483        | 0.0193607         | $\bar{1}.9488374$ | 0.0351249         |
| .33 | 2.408435   | .4325214        | 0.0025427         | $\bar{1}.9542989$ | 0.0695813         |
| .35 | 2.469180   | .4058835        | $\bar{1}.9883379$ | $\bar{1}.9633451$ | 0.1086357         |
| .37 | 2.532230   | .3808532        | $\bar{1}.9765313$ | $\bar{1}.9757126$ | 0.1519704         |
| .39 | 2.597600   | .3572771        | $\bar{1}.9669390$ | $\bar{1}.9911732$ | 0.1993083         |
| .41 | 2.665316   | .3350245        | $\bar{1}.9594015$ | 0.0095272         | 0.2504056         |
| .43 | 2.735407   | .3139830        | $\bar{1}.9537798$ | 0.0305985         | 0.3050462         |
| .45 | 2.807911   | .2940552        | $\bar{1}.9499515$ | 0.0542311         | 0.3630370         |
| .47 | 2.882866   | .2751560        | $\bar{1}.9478084$ | 0.0802853         | 0.4242050         |
| .49 | 2.960320   | .2572107        | $\bar{1}.9472539$ | 0.1086357         | 0.4883938         |
| .50 | 3.000000   | .2485749        | $\bar{1}.9475449$ | 0.1236362         | 0.5215762         |
| .51 | 3.040322   | .2401535        | $\bar{1}.9482015$ | 0.1391691         | 0.5554617         |
| .53 | 3.122927   | .2239256        | $\bar{1}.9505733$ | 0.1717828         | 0.6252795         |
| .55 | 3.208190   | .2084747        | $\bar{1}.9542989$ | 0.2063831         | 0.6977292         |
| .57 | 3.296173   | .1937540        | $\bar{1}.9593141$ | 0.2428843         | 0.7727022         |
| .59 | 3.386943   | .1797213        | $\bar{1}.9655606$ | 0.2812078         | 0.8500985         |
| .61 | 3.480565   | .1663382        | $\bar{1}.9729848$ | 0.3212811         | 0.9298261         |
| .63 | 3.577111   | .1535702        | $\bar{1}.9815374$ | 0.3630370         | 1.0117989         |
| .65 | 3.676653   | .1413855        | $\bar{1}.9911732$ | 0.4064136         | 1.0959377         |
| .67 | 3.779275   | .1297555        | 0.0018501         | 0.4513532         | 1.1821682         |
| .69 | 3.885053   | .1186537        | 0.0135293         | 0.4978018         | 1.2704211         |
| .70 | 3.939151   | .1132932        | 0.0197334         | 0.5215762         | 1.3152855         |

TABLE II

REFERENCE TABLE FOR GENERALIZED HYPERBOLIC-SECANT CURVE

| $m$ | $\alpha_1$ | $\gamma \cdot \sigma$ | $\kappa \cdot \sigma$ | $\kappa/\gamma$ |
|-----|------------|-----------------------|-----------------------|-----------------|
| 1   | 5.00000    | 1.57080               | .5000                 | .3183099        |
| 2   | 4.20000    | .90690                | .4535                 | .5000000        |
| 3   | 3.80625    | .68367                | .4352                 | .6366198        |
| 4   | 3.59376    | .56786                | .4259                 | .7500000        |
| 5   | 3.46560    | .49516                | .4203                 | .8488264        |
| 6   | 3.38128    | .44437                | .4166                 | .9375000        |
| 7   | 3.32210    | .40642                | .4140                 | 1.0185916       |
| 8   | 3.27847    | .37671                | .4120                 | 1.0937500       |
| 9   | 3.24507    | .35265                | .4105                 | 1.1641047       |
| 10  | 3.21872    | .33266                | .4093                 | 1.2304688       |
| 11  | 3.19743    | .31571                | .4084                 | 1.2934497       |
| 12  | 3.17987    | .30110                | .4075                 | 1.3535156       |
| 13  | 3.16516    | .28834                | .4069                 | 1.4110360       |
| 14  | 3.15266    | .27708                | .4063                 | 1.4663086       |
| 15  | 3.14190    | .26704                | .4058                 | 1.5195773       |
| 16  | 3.13256    | .25801                | .4053                 | 1.5710448       |
| 17  | 3.12436    | .24984                | .4050                 | 1.6208824       |
| 18  | 3.11711    | .24240                | .4046                 | 1.6692351       |
| 19  | 3.11066    | .23558                | .4043                 | 1.7162284       |
| 20  | 3.10488    | .22931                | .4040                 | 1.7619704       |
| 21  | 3.09967    | .22351                | .4038                 | 1.8065563       |
| 22  | 3.09495    | .21814                | .4036                 | 1.8500689       |
| 23  | 3.09066    | .21313                | .4034                 | 1.8925828       |
| 24  | 3.08674    | .20845                | .4032                 | 1.9341630       |
| 25  | 3.08314    | .20407                | .4030                 | 1.9748689       |
| 26  | 3.07983    | .19995                | .4028                 | 2.0147531       |
| 27  | 3.07677    | .19607                | .4027                 | 2.0538637       |
| 28  | 3.07394    | .19241                | .4026                 | 2.0922436       |
| 29  | 3.07131    | .18894                | .4024                 | 2.1299327       |
| 30  | 3.06886    | .18566                | .4023                 | 2.1669666       |
| 31  | 3.06657    | .18254                | .4022                 | 2.2033787       |
| 32  | 3.06443    | .17957                | .4021                 | 2.2391988       |
| 33  | 3.06242    | .17675                | .4020                 | 2.2744554       |
| 34  | 3.06053    | .17405                | .4019                 | 2.3081738       |
| 35  | 3.05876    | .17147                | .4018                 | 2.3433783       |
| 36  | 3.05708    | .16901                | .4018                 | 2.3770906       |
| 37  | 3.05550    | .16665                | .4017                 | 2.4103320       |
| 38  | 3.05400    | .16438                | .4016                 | 2.4431209       |
| 39  | 3.05258    | .16220                | .4015                 | 2.4754761       |
| 40  | 3.05124    | .16011                | .4015                 | 2.5074136       |
| 41  | 3.04996    | .15810                | .4014                 | 2.5389498       |
| 42  | 3.04874    | .15616                | .4013                 | 2.5700989       |
| 43  | 3.04758    | .15429                | .4013                 | 2.6008754       |
| 44  | 3.04648    | .15248                | .4012                 | 2.6312918       |
| 45  | 3.04542    | .15074                | .4012                 | 2.6613609       |

TABLE II. *Continued*

| $m$ | $\alpha_1$ | $\gamma \cdot \sigma$ | $\kappa \cdot \sigma$ | $\kappa/\gamma$ |
|-----|------------|-----------------------|-----------------------|-----------------|
| 46  | 3.04442    | .14906                | .4011                 | 2.6910938       |
| 47  | 3.04345    | .14743                | .4011                 | 2.7205023       |
| 48  | 3.04253    | .14585                | .4010                 | 2.7495959       |
| 49  | 3.04164    | .14433                | .4010                 | 2.7783853       |
| 50  | 3.04079    | .14285                | .4010                 | 2.8068791       |
| 51  | 3.03998    | .14144                | .4009                 | 2.8350870       |
| 52  | 3.03920    | .14002                | .4009                 | 2.8630167       |
| 53  | 3.03844    | .13867                | .4009                 | 2.8906770       |
| 54  | 3.03772    | .13735                | .4008                 | 2.9180747       |
| 55  | 3.03702    | .13607                | .4008                 | 2.9452180       |
| 56  | 3.03635    | .13483                | .4007                 | 2.9721131       |
| 57  | 3.03570    | .13362                | .4007                 | 2.9987674       |
| 58  | 3.03507    | .13245                | .4007                 | 3.0251866       |
| 59  | 3.03447    | .13130                | .4006                 | 3.0513774       |
| 60  | 3.03389    | .13018                | .4006                 | 3.0773450       |
| 61  | 3.03332    | .12909                | .4006                 | 3.1030957       |
| 62  | 3.03278    | .12803                | .4006                 | 3.1286341       |
| 63  | 3.03225    | .12699                | .4005                 | 3.1539661       |
| 64  | 3.03174    | .12598                | .4005                 | 3.1790959       |
| 65  | 3.03124    | .12499                | .4005                 | 3.2040290       |
| 66  | 3.03076    | .12403                | .4005                 | 3.2287693       |
| 67  | 3.03030    | .12268                | .4004                 | 3.2533218       |
| 68  | 3.02984    | .12216                | .4004                 | 3.2776900       |
| 69  | 3.02940    | .12126                | .4004                 | 3.3018788       |
| 70  | 3.02898    | .12038                | .4004                 | 3.3258913       |
| 71  | 3.02856    | .11952                | .4004                 | 3.3497321       |
| 72  | 3.02816    | .11867                | .4003                 | 3.3734040       |
| 73  | 3.02777    | .11785                | .4003                 | 3.3969115       |
| 74  | 3.02739    | .11704                | .4003                 | 3.4202569       |
| 75  | 3.02702    | .11624                | .4003                 | 3.4434445       |
| 76  | 3.02666    | .11547                | .4003                 | 3.4664766       |
| 77  | 3.02631    | .11470                | .4002                 | 3.4893571       |
| 78  | 3.02597    | .11396                | .4002                 | 3.5120881       |
| 79  | 3.02564    | .11322                | .4002                 | 3.5346734       |
| 80  | 3.02531    | .11251                | .4002                 | 3.5571148       |
| 81  | 3.02499    | .11180                | .4002                 | 3.5794161       |
| 82  | 3.02469    | .11111                | .4002                 | 3.6015788       |
| 83  | 3.02439    | .11043                | .4002                 | 3.6236064       |
| 84  | 3.02409    | .10976                | .4001                 | 3.6455005       |
| 85  | 3.02380    | .10911                | .4001                 | 3.6672643       |
| 86  | 3.02352    | .10846                | .4001                 | 3.6888993       |
| 87  | 3.02326    | .10783                | .4001                 | 3.7104086       |
| 88  | 3.02298    | .10721                | .4001                 | 3.7317935       |
| 89  | 3.02272    | .10660                | .4001                 | 3.7530570       |
| 90  | 3.02247    | .10600                | .4001                 | 3.7742002       |

TABLE II. *Continued*

| $m$ | $\alpha_4$ | $\gamma' \sigma$ | $\kappa' \sigma$ | $\kappa/\gamma$ |
|-----|------------|------------------|------------------|-----------------|
| 91  | 3.02222    | .10541           | .4001            | 3.7952262       |
| 92  | 3.02197    | .10483           | .4001            | 3.8161358       |
| 93  | 3.02174    | .10426           | .4001            | 3.8369320       |
| 94  | 3.02150    | .10369           | .4001            | 3.8576155       |
| 95  | 3.02127    | .10314           | .4001            | 3.8781893       |
| 96  | 3.02105    | .10260           | .4001            | 3.8986540       |
| 97  | 3.02083    | .10206           | .4001            | 3.9190124       |
| 98  | 3.02062    | .10153           | .4000            | 3.9392650       |
| 99  | 3.02040    | .10101           | .4000            | 3.9594146       |
| 100 | 3.02020    | .10050           | .3999            | 3.9794615       |



TABLE III

$$\sum_{s=1}^n \frac{1}{s^r}$$

| $n$ | $r = 2$   | $r = 4$   | $r = 6$   |
|-----|-----------|-----------|-----------|
| 1   | 1.0000000 | 1.0000000 | 1.0000000 |
| 2   | 1.2500000 | 1.0625000 | 1.0156250 |
| 3   | 1.3611111 | 1.0748457 | 1.0169967 |
| 4   | 1.4236111 | 1.0787519 | 1.0172409 |
| 5   | 1.4636111 | 1.0803519 | 1.0173049 |
| 6   | 1.4913889 | 1.0811235 | 1.0173263 |
| 7   | 1.5117971 | 1.0815400 | 1.0173348 |
| 8   | 1.5274221 | 1.0817842 | 1.0173386 |
| 9   | 1.5397677 | 1.0819366 | 1.0173405 |
| 10  | 1.5497677 | 1.0820366 | 1.0173415 |
| 11  | 1.5580322 | 1.0821049 | 1.0173421 |
| 12  | 1.5649766 | 1.0821531 | 1.0173424 |
| 13  | 1.5708938 | 1.0821881 | 1.0173426 |
| 14  | 1.5759958 | 1.0822142 | 1.0173428 |
| 15  | 1.5804403 | 1.0822339 | 1.0173428 |
| 16  | 1.5843465 | 1.0822492 | 1.0173429 |
| 17  | 1.5878067 | 1.0822611 | 1.0173429 |
| 18  | 1.5908932 | 1.0822707 | 1.0173430 |
| 19  | 1.5936632 | 1.0822783 | 1.0173430 |
| 20  | 1.5961632 | 1.0822846 | 1.0173430 |
| 21  | 1.5984308 | 1.0822897 | 1.0173430 |
| 22  | 1.6004969 | 1.0822940 | 1.0173430 |
| 23  | 1.6023873 | 1.0822976 | 1.0173430 |
| 24  | 1.6041234 | 1.0823006 | 1.0173430 |
| 25  | 1.6057234 | 1.0823031 | 1.0173430 |
| 26  | 1.6072027 | 1.0823053 | 1.0173430 |
| 27  | 1.6085744 | 1.0823072 | 1.0173430 |
| 28  | 1.6098499 | 1.0823088 | 1.0173431 |
| 29  | 1.6110390 | 1.0823103 | 1.0173431 |
| 30  | 1.6121501 | 1.0823115 | 1.0173431 |
| 31  | 1.6131907 | 1.0823126 | 1.0173431 |
| 32  | 1.6141673 | 1.0823135 | 1.0173431 |
| 33  | 1.6150855 | 1.0823144 | 1.0173431 |
| 34  | 1.6159506 | 1.0823151 | 1.0173431 |
| 35  | 1.6167669 | 1.0823158 | 1.0173431 |
| 36  | 1.6175385 | 1.0823164 | 1.0173431 |
| 37  | 1.6182690 | 1.0823169 | 1.0173431 |
| 38  | 1.6189615 | 1.0823174 | 1.0173431 |
| 39  | 1.6196190 | 1.0823178 | 1.0173431 |
| 40  | 1.6202440 | 1.0823182 | 1.0173431 |

TABLE III. *Continued*

| $n$ | $r = 2$   | $r = 4$   | $r = 6$   |
|-----|-----------|-----------|-----------|
| 41  | 1.6208388 | 1.0823186 | 1.0173431 |
| 42  | 1.6214057 | 1.0823189 | 1.0173431 |
| 43  | 1.6219466 | 1.0823192 | 1.0173431 |
| 44  | 1.6224631 | 1.0823195 | 1.0173431 |
| 45  | 1.6229569 | 1.0823197 | 1.0173431 |
| 46  | 1.6234295 | 1.0823199 | 1.0173431 |
| 47  | 1.6238822 | 1.0823201 | 1.0173431 |
| 48  | 1.6243162 | 1.0823203 | 1.0173431 |
| 49  | 1.6247327 | 1.0823205 | 1.0173431 |
| 50  | 1.6251327 | 1.0823206 | 1.0173431 |
| 51  | 1.6255172 | 1.0823208 | 1.0173431 |
| 52  | 1.6258870 | 1.0823209 | 1.0173431 |
| 53  | 1.6262430 | 1.0823211 | 1.0173431 |
| 54  | 1.6265860 | 1.0823212 | 1.0173431 |
| 55  | 1.6269165 | 1.0823213 | 1.0173431 |
| 56  | 1.6272354 | 1.0823214 | 1.0173431 |
| 57  | 1.6275432 | 1.0823215 | 1.0173431 |
| 58  | 1.6278405 | 1.0823216 | 1.0173431 |
| 59  | 1.6281277 | 1.0823217 | 1.0173431 |
| 60  | 1.6284055 | 1.0823217 | 1.0173431 |
| 61  | 1.6286743 | 1.0823218 | 1.0173431 |
| 62  | 1.6289344 | 1.0823219 | 1.0173431 |
| 63  | 1.6291864 | 1.0823219 | 1.0173431 |
| 64  | 1.6294305 | 1.0823220 | 1.0173431 |
| 65  | 1.6296672 | 1.0823220 | 1.0173431 |
| 66  | 1.6298968 | 1.0823221 | 1.0173431 |
| 67  | 1.6301195 | 1.0823222 | 1.0173431 |
| 68  | 1.6303358 | 1.0823222 | 1.0173431 |
| 69  | 1.6305458 | 1.0823222 | 1.0173431 |
| 70  | 1.6307499 | 1.0823223 | 1.0173431 |
| 71  | 1.6309483 | 1.0823223 | 1.0173431 |
| 72  | 1.6311412 | 1.0823224 | 1.0173431 |
| 73  | 1.6313288 | 1.0823224 | 1.0173431 |
| 74  | 1.6315114 | 1.0823224 | 1.0173431 |
| 75  | 1.6316892 | 1.0823225 | 1.0173431 |
| 76  | 1.6318624 | 1.0823225 | 1.0173431 |
| 77  | 1.6320310 | 1.0823225 | 1.0173431 |
| 78  | 1.6321954 | 1.0823225 | 1.0173431 |
| 79  | 1.6323556 | 1.0823226 | 1.0173431 |
| 80  | 1.6325119 | 1.0823226 | 1.0173431 |
| 81  | 1.6326643 | 1.0823226 | 1.0173431 |
| 82  | 1.6328130 | 1.0823226 | 1.0173431 |
| 83  | 1.6329582 | 1.0823227 | 1.0173431 |
| 84  | 1.6330999 | 1.0823227 | 1.0173431 |
| 85  | 1.6332383 | 1.0823227 | 1.0173431 |

TABLE III. *Continued*

| $n$ | $r = 2$   | $r = 4$   | $r = 6$   |
|-----|-----------|-----------|-----------|
| 86  | 1.6333735 | 1.0823227 | 1.0173431 |
| 87  | 1.6335056 | 1.0823227 | 1.0173431 |
| 88  | 1.6336348 | 1.0823228 | 1.0173431 |
| 89  | 1.6337610 | 1.0823228 | 1.0173431 |
| 90  | 1.6338845 | 1.0823228 | 1.0173431 |
| 91  | 1.6340052 | 1.0823228 | 1.0173431 |
| 92  | 1.6341234 | 1.0823228 | 1.0173431 |
| 93  | 1.6342390 | 1.0823228 | 1.0173431 |
| 94  | 1.6343522 | 1.0823228 | 1.0173431 |
| 95  | 1.6344630 | 1.0823229 | 1.0173431 |
| 96  | 1.6345715 | 1.0823229 | 1.0173431 |
| 97  | 1.6346777 | 1.0823229 | 1.0173431 |
| 98  | 1.6347819 | 1.0823229 | 1.0173431 |
| 99  | 1.6348839 | 1.0823229 | 1.0173431 |
| 100 | 1.6349839 | 1.0823229 | 1.0173431 |

TABLE IV

$$\sum_{s=0}^n \frac{1}{(2s+1)^r}$$

| $n$ | $r = 2$   | $r = 4$   | $r = 6$   |
|-----|-----------|-----------|-----------|
| 1   | 1.1111111 | 1.0123457 | 1.0013717 |
| 2   | 1.1511111 | 1.0139457 | 1.0014357 |
| 3   | 1.1715193 | 1.0143622 | 1.0014442 |
| 4   | 1.1838650 | 1.0145146 | 1.0014461 |
| 5   | 1.1921294 | 1.0145829 | 1.0014467 |
| 6   | 1.1980466 | 1.0146179 | 1.0014469 |
| 7   | 1.2024910 | 1.0146377 | 1.0014470 |
| 8   | 1.2059512 | 1.0146496 | 1.0014470 |
| 9   | 1.2087213 | 1.0146573 | 1.0014470 |
| 10  | 1.2109889 | 1.0146624 | 1.0014471 |
| 11  | 1.2128792 | 1.0146660 | 1.0014471 |
| 12  | 1.2144792 | 1.0146686 | 1.0014471 |
| 13  | 1.2158510 | 1.0146705 | 1.0014471 |
| 14  | 1.2170400 | 1.0146719 | 1.0014471 |
| 15  | 1.2180806 | 1.0146730 | 1.0014471 |
| 16  | 1.2189989 | 1.0146738 | 1.0014471 |
| 17  | 1.2198152 | 1.0146745 | 1.0014471 |
| 18  | 1.2205457 | 1.0146750 | 1.0014471 |
| 19  | 1.2212032 | 1.0146754 | 1.0014471 |
| 20  | 1.2217980 | 1.0146758 | 1.0014471 |
| 21  | 1.2223389 | 1.0146761 | 1.0014471 |
| 22  | 1.2228327 | 1.0146763 | 1.0014471 |
| 23  | 1.2232854 | 1.0146765 | 1.0014471 |
| 24  | 1.2237019 | 1.0146767 | 1.0014471 |
| 25  | 1.2240864 | 1.0146768 | 1.0014471 |
| 26  | 1.2244423 | 1.0146770 | 1.0014471 |
| 27  | 1.2247729 | 1.0146771 | 1.0014471 |
| 28  | 1.2250807 | 1.0146772 | 1.0014471 |
| 29  | 1.2253680 | 1.0146773 | 1.0014471 |
| 30  | 1.2256367 | 1.0146773 | 1.0014471 |
| 31  | 1.2258887 | 1.0146774 | 1.0014471 |
| 32  | 1.2261254 | 1.0146775 | 1.0014471 |
| 33  | 1.2263481 | 1.0146775 | 1.0014471 |
| 34  | 1.2265582 | 1.0146775 | 1.0014471 |
| 35  | 1.2267566 | 1.0146776 | 1.0014471 |
| 36  | 1.2269442 | 1.0146776 | 1.0014471 |
| 37  | 1.2271220 | 1.0146777 | 1.0014471 |
| 38  | 1.2272906 | 1.0146777 | 1.0014471 |
| 39  | 1.2274509 | 1.0146777 | 1.0014471 |
| 40  | 1.2276033 | 1.0146777 | 1.0014471 |

TABLE IV. *Continued*

| $n$ | $r = 2$   | $r = 4$   | $r = 6$   |
|-----|-----------|-----------|-----------|
| 41  | 1.2277485 | 1.0146778 | 1.0014471 |
| 42  | 1.2278869 | 1.0146778 | 1.0014471 |
| 43  | 1.2280190 | 1.0146778 | 1.0014471 |
| 44  | 1.2281452 | 1.0146778 | 1.0014471 |
| 45  | 1.2282660 | 1.0146778 | 1.0014471 |
| 46  | 1.2283816 | 1.0146778 | 1.0014471 |
| 47  | 1.2284924 | 1.0146778 | 1.0014471 |
| 48  | 1.2285987 | 1.0146779 | 1.0014471 |
| 49  | 1.2287007 | 1.0146779 | 1.0014471 |

TABLE V

AREA OF HYPERBOLIC-SECANT CURVE IN TERMS OF THE ABCISSA

| $t$ | $A(-t)$ | $\Delta A(-t)$ | $t$ | $A(-t)$ | $\Delta A(-t)$ |
|-----|---------|----------------|-----|---------|----------------|
| .00 | .500000 | .....          | .45 | .361364 | .002891        |
| .01 | .496817 | .003183        | .46 | .358483 | .002881        |
| .02 | .493634 | .003183        | .47 | .355616 | .002867        |
| .03 | .490452 | .003182        | .48 | .352761 | .002855        |
| .04 | .487271 | .003181        | .49 | .349919 | .002842        |
| .05 | .484091 | .003180        | .50 | .347090 | .002829        |
| .06 | .480913 | .003178        | .51 | .344273 | .002817        |
| .07 | .477737 | .003176        | .52 | .341470 | .002803        |
| .08 | .474562 | .003175        | .53 | .338681 | .002789        |
| .09 | .471391 | .003171        | .54 | .335905 | .002776        |
| .10 | .468222 | .003169        | .55 | .333142 | .002763        |
| .11 | .465056 | .003166        | .56 | .330394 | .002748        |
| .12 | .461894 | .003162        | .57 | .327658 | .002736        |
| .13 | .458735 | .003159        | .58 | .324938 | .002720        |
| .14 | .455582 | .003153        | .59 | .322231 | .002707        |
| .15 | .452431 | .003151        | .60 | .319539 | .002692        |
| .16 | .449286 | .003145        | .61 | .316861 | .002678        |
| .17 | .446146 | .003140        | .62 | .314197 | .002664        |
| .18 | .443011 | .003135        | .63 | .311548 | .002649        |
| .19 | .439882 | .003129        | .64 | .308915 | .002633        |
| .20 | .436758 | .003124        | .65 | .306296 | .002619        |
| .21 | .433641 | .003117        | .66 | .303692 | .002604        |
| .22 | .430529 | .003112        | .67 | .301103 | .002589        |
| .23 | .427426 | .003103        | .68 | .298529 | .002574        |
| .24 | .424328 | .003098        | .69 | .295970 | .002559        |
| .25 | .421239 | .003089        | .70 | .293426 | .002544        |
| .26 | .418156 | .003083        | .71 | .290897 | .002529        |
| .27 | .415082 | .003074        | .72 | .288385 | .002512        |
| .28 | .412015 | .003067        | .73 | .285887 | .002498        |
| .29 | .408957 | .003058        | .74 | .283405 | .002482        |
| .30 | .405907 | .003050        | .75 | .280939 | .002466        |
| .31 | .402868 | .003039        | .76 | .278488 | .002451        |
| .32 | .399836 | .003032        | .77 | .276053 | .002435        |
| .33 | .396814 | .003022        | .78 | .273633 | .002420        |
| .34 | .393801 | .003013        | .79 | .271229 | .002404        |
| .35 | .390799 | .003002        | .80 | .268841 | .002388        |
| .36 | .387806 | .002993        | .81 | .266470 | .002371        |
| .37 | .384825 | .002981        | .82 | .264114 | .002356        |
| .38 | .381854 | .002971        | .83 | .261773 | .002341        |
| .39 | .378891 | .002963        | .84 | .259449 | .002324        |
| .40 | .375941 | .002950        | .85 | .257140 | .002309        |
| .41 | .373003 | .002938        | .86 | .254847 | .002293        |
| .42 | .370075 | .002928        | .87 | .252570 | .002277        |
| .43 | .367160 | .002915        | .88 | .250309 | .002261        |
| .44 | .364255 | .002905        | .89 | .248064 | .002245        |

TABLE V. *Continued*

| $t$  | $A(-t)$ | $\Delta A(-t)$ | $t$  | $A(-t)$ | $\Delta A(-t)$ |
|------|---------|----------------|------|---------|----------------|
| .90  | .245835 | .002229        | 1.35 | .161483 | .001553        |
| .91  | .243622 | .002213        | 1.36 | .159943 | .001540        |
| .92  | .241425 | .002197        | 1.37 | .158417 | .001526        |
| .93  | .239243 | .002182        | 1.38 | .156904 | .001513        |
| .94  | .237078 | .002165        | 1.39 | .155404 | .001500        |
| .95  | .234928 | .002150        | 1.40 | .153918 | .001486        |
| .96  | .232794 | .002134        | 1.41 | .152444 | .001474        |
| .97  | .230676 | .002118        | 1.42 | .150984 | .001460        |
| .98  | .228574 | .002102        | 1.43 | .149537 | .001447        |
| .99  | .226488 | .002086        | 1.44 | .148102 | .001435        |
| 1.00 | .224417 | .002071        | 1.45 | .146680 | .001422        |
| 1.01 | .222362 | .002055        | 1.46 | .145271 | .001409        |
| 1.02 | .220323 | .002039        | 1.47 | .143875 | .001396        |
| 1.03 | .218299 | .002024        | 1.48 | .142491 | .001384        |
| 1.04 | .216291 | .002008        | 1.49 | .141119 | .001372        |
| 1.05 | .214299 | .001992        | 1.50 | .139760 | .001359        |
| 1.06 | .212322 | .001977        | 1.51 | .138413 | .001347        |
| 1.07 | .210360 | .001962        | 1.52 | .137078 | .001335        |
| 1.08 | .208414 | .001946        | 1.53 | .135755 | .001323        |
| 1.09 | .206483 | .001931        | 1.54 | .134444 | .001311        |
| 1.10 | .204568 | .001915        | 1.55 | .133145 | .001299        |
| 1.11 | .202668 | .001900        | 1.56 | .131858 | .001287        |
| 1.12 | .200783 | .001885        | 1.57 | .130583 | .001275        |
| 1.13 | .198913 | .001870        | 1.58 | .129319 | .001264        |
| 1.14 | .197059 | .001854        | 1.59 | .128067 | .001252        |
| 1.15 | .195219 | .001840        | 1.60 | .126826 | .001241        |
| 1.16 | .193394 | .001825        | 1.61 | .125597 | .001229        |
| 1.17 | .191585 | .001809        | 1.62 | .124379 | .001218        |
| 1.18 | .189790 | .001795        | 1.63 | .123172 | .001207        |
| 1.19 | .188010 | .001780        | 1.64 | .121977 | .001195        |
| 1.20 | .186245 | .001765        | 1.65 | .120792 | .001185        |
| 1.21 | .184494 | .001751        | 1.66 | .119618 | .001174        |
| 1.22 | .182758 | .001736        | 1.67 | .118456 | .001162        |
| 1.23 | .181036 | .001722        | 1.68 | .117304 | .001152        |
| 1.24 | .179329 | .001707        | 1.69 | .116162 | .001142        |
| 1.25 | .177636 | .001693        | 1.70 | .115032 | .001130        |
| 1.26 | .175958 | .001678        | 1.71 | .113911 | .001121        |
| 1.27 | .174294 | .001664        | 1.72 | .112802 | .001109        |
| 1.28 | .172644 | .001650        | 1.73 | .111702 | .001100        |
| 1.29 | .171008 | .001636        | 1.74 | .110613 | .001089        |
| 1.30 | .169385 | .001623        | 1.75 | .109534 | .001079        |
| 1.31 | .167777 | .001608        | 1.76 | .108465 | .001069        |
| 1.32 | .166183 | .001594        | 1.77 | .107407 | .001058        |
| 1.33 | .164603 | .001580        | 1.78 | .106358 | .001049        |
| 1.34 | .163036 | .001567        | 1.79 | .105319 | .001039        |



TABLE V. *Continued*

| $t$  | $A(-t)$ | $\Delta A(-t)$ | $t$  | $A(-t)$ | $\Delta A(-t)$ |
|------|---------|----------------|------|---------|----------------|
| 1.80 | .104290 | .001029        | 2.25 | .063852 | .000367        |
| 1.81 | .103270 | .001020        | 2.26 | .066192 | .000660        |
| 1.82 | .102260 | .001010        | 2.27 | .065538 | .000654        |
| 1.83 | .101260 | .001000        | 2.28 | .064891 | .000647        |
| 1.84 | .100269 | .000991        | 2.29 | .064249 | .000642        |
| 1.85 | .099287 | .000982        | 2.30 | .063614 | .000635        |
| 1.86 | .098315 | .000972        | 2.31 | .062985 | .000629        |
| 1.87 | .097352 | .000963        | 2.32 | .062363 | .000622        |
| 1.88 | .096398 | .000954        | 2.33 | .061746 | .000617        |
| 1.89 | .095453 | .000945        | 2.34 | .061136 | .000610        |
| 1.90 | .094518 | .000935        | 2.35 | .060531 | .000605        |
| 1.91 | .093591 | .000927        | 2.36 | .059932 | .000599        |
| 1.92 | .092673 | .000918        | 2.37 | .059339 | .000593        |
| 1.93 | .091763 | .000910        | 2.38 | .058752 | .000587        |
| 1.94 | .090863 | .000900        | 2.39 | .058171 | .000581        |
| 1.95 | .089971 | .000892        | 2.40 | .057595 | .000576        |
| 1.96 | .089087 | .000884        | 2.41 | .057025 | .000570        |
| 1.97 | .088212 | .000875        | 2.42 | .056461 | .000564        |
| 1.98 | .087345 | .000867        | 2.43 | .055902 | .000559        |
| 1.99 | .086487 | .000858        | 2.44 | .055349 | .000553        |
| 2.00 | .085637 | .000850        | 2.45 | .054801 | .000548        |
| 2.01 | .084795 | .000842        | 2.46 | .054258 | .000543        |
| 2.02 | .083961 | .000834        | 2.47 | .053721 | .000537        |
| 2.03 | .083135 | .000826        | 2.48 | .053188 | .000533        |
| 2.04 | .082317 | .000818        | 2.49 | .052662 | .000526        |
| 2.05 | .081507 | .000810        | 2.50 | .052140 | .000522        |
| 2.06 | .080705 | .000802        | 2.51 | .051624 | .000516        |
| 2.07 | .079910 | .000795        | 2.52 | .051112 | .000512        |
| 2.08 | .079123 | .000787        | 2.53 | .050606 | .000506        |
| 2.09 | .078344 | .000779        | 2.54 | .050104 | .000502        |
| 2.10 | .077572 | .000772        | 2.55 | .049608 | .000496        |
| 2.11 | .076808 | .000764        | 2.56 | .049116 | .000492        |
| 2.12 | .076051 | .000757        | 2.57 | .048629 | .000487        |
| 2.13 | .075301 | .000750        | 2.58 | .048147 | .000482        |
| 2.14 | .074559 | .000742        | 2.59 | .047670 | .000477        |
| 2.15 | .073824 | .000735        | 2.60 | .047197 | .000473        |
| 2.16 | .073095 | .000729        | 2.61 | .046729 | .000468        |
| 2.17 | .072374 | .000721        | 2.62 | .046266 | .000463        |
| 2.18 | .071660 | .000714        | 2.63 | .045807 | .000459        |
| 2.19 | .070953 | .000707        | 2.64 | .045353 | .000454        |
| 2.20 | .070253 | .000700        | 2.65 | .044903 | .000450        |
| 2.21 | .069559 | .000694        | 2.66 | .044458 | .000445        |
| 2.22 | .068873 | .000686        | 2.67 | .044017 | .000441        |
| 2.23 | .068193 | .000680        | 2.68 | .043581 | .000436        |
| 2.24 | .067519 | .000674        | 2.69 | .043148 | .000433        |

TABLE V. *Continued*

| $t$  | $A(-t)$ | $\Delta A(-t)$ | $t$  | $A(-t)$ | $\Delta A(-t)$ |
|------|---------|----------------|------|---------|----------------|
| 2.70 | .042720 | .000428        | 3.15 | .027264 | .000274        |
| 2.71 | .042296 | .000424        | 3.16 | .026993 | .000271        |
| 2.72 | .041877 | .000419        | 3.17 | .026725 | .000268        |
| 2.73 | .041461 | .000416        | 3.18 | .026459 | .000266        |
| 2.74 | .041050 | .000411        | 3.19 | .026196 | .000263        |
| 2.75 | .040642 | .000408        | 3.20 | .025936 | .000260        |
| 2.76 | .040239 | .000403        | 3.21 | .025678 | .000258        |
| 2.77 | .039840 | .000399        | 3.22 | .025423 | .000255        |
| 2.78 | .039444 | .000396        | 3.23 | .025170 | .000253        |
| 2.79 | .039053 | .000391        | 3.24 | .024920 | .000250        |
| 2.80 | .038665 | .000388        | 3.25 | .024672 | .000248        |
| 2.81 | .038282 | .000383        | 3.26 | .024427 | .000245        |
| 2.82 | .037902 | .000380        | 3.27 | .024184 | .000243        |
| 2.83 | .037525 | .000377        | 3.28 | .023944 | .000240        |
| 2.84 | .037153 | .000372        | 3.29 | .023706 | .000238        |
| 2.85 | .036784 | .000369        | 3.30 | .023470 | .000236        |
| 2.86 | .036419 | .000365        | 3.31 | .023237 | .000233        |
| 2.87 | .036057 | .000362        | 3.32 | .023006 | .000231        |
| 2.88 | .035699 | .000358        | 3.33 | .022777 | .000229        |
| 2.89 | .035345 | .000354        | 3.34 | .022551 | .000226        |
| 2.90 | .034994 | .000351        | 3.35 | .022326 | .000225        |
| 2.91 | .034646 | .000348        | 3.36 | .022104 | .000222        |
| 2.92 | .034302 | .000344        | 3.37 | .021884 | .000220        |
| 2.93 | .033961 | .000341        | 3.38 | .021667 | .000217        |
| 2.94 | .033624 | .000337        | 3.39 | .021452 | .000215        |
| 2.95 | .033290 | .000334        | 3.40 | .021238 | .000214        |
| 2.96 | .032960 | .000330        | 3.41 | .021027 | .000211        |
| 2.97 | .032632 | .000328        | 3.42 | .020818 | .000209        |
| 2.98 | .032308 | .000324        | 3.43 | .020611 | .000207        |
| 2.99 | .031987 | .000321        | 3.44 | .020406 | .000205        |
| 3.00 | .031669 | .000318        | 3.45 | .020203 | .000203        |
| 3.01 | .031355 | .000314        | 3.46 | .020002 | .000201        |
| 3.02 | .031043 | .000312        | 3.47 | .019803 | .000199        |
| 3.03 | .030735 | .000308        | 3.48 | .019606 | .000197        |
| 3.04 | .030430 | .000305        | 3.49 | .019411 | .000195        |
| 3.05 | .030127 | .000303        | 3.50 | .019218 | .000193        |
| 3.06 | .029828 | .000299        | 3.51 | .019027 | .000191        |
| 3.07 | .029532 | .000296        | 3.52 | .018838 | .000189        |
| 3.08 | .029238 | .000294        | 3.53 | .018651 | .000187        |
| 3.09 | .028948 | .000290        | 3.54 | .018465 | .000186        |
| 3.10 | .028660 | .000288        | 3.55 | .018282 | .000183        |
| 3.11 | .028375 | .000285        | 3.56 | .018100 | .000182        |
| 3.12 | .028093 | .000282        | 3.57 | .017920 | .000180        |
| 3.13 | .027814 | .000279        | 3.58 | .017742 | .000178        |
| 3.14 | .027538 | .000276        | 3.59 | .017565 | .000177        |

TABLE V. *Continued*

| $t$  | $A(-t)$ | $\Delta A(-t)$ | $t$  | $A(-t)$ | $\Delta A(-t)$ |
|------|---------|----------------|------|---------|----------------|
| 3.60 | .017391 | .000174        | 4.05 | .011090 | .000112        |
| 3.61 | .017218 | .000173        | 4.06 | .010980 | .000110        |
| 3.62 | .017046 | .000172        | 4.07 | .010871 | .000109        |
| 3.63 | .016877 | .000169        | 4.08 | .010763 | .000108        |
| 3.64 | .016709 | .000168        | 4.09 | .010656 | .000107        |
| 3.65 | .016543 | .000166        | 4.10 | .010550 | .000106        |
| 3.66 | .016378 | .000165        | 4.11 | .010445 | .000105        |
| 3.67 | .016215 | .000163        | 4.12 | .010341 | .000104        |
| 3.68 | .016054 | .000161        | 4.13 | .010238 | .000103        |
| 3.69 | .015894 | .000160        | 4.14 | .010136 | .000102        |
| 3.70 | .015736 | .000158        | 4.15 | .010035 | .000101        |
| 3.71 | .015580 | .000156        | 4.16 | .009935 | .000100        |
| 3.72 | .015425 | .000155        | 4.17 | .009837 | .000098        |
| 3.73 | .015271 | .000154        | 4.18 | .009739 | .000098        |
| 3.74 | .015120 | .000151        | 4.19 | .009642 | .000097        |
| 3.75 | .014969 | .000151        | 4.20 | .009546 | .000096        |
| 3.76 | .014820 | .000149        | 4.21 | .009451 | .000095        |
| 3.77 | .014673 | .000147        | 4.22 | .009357 | .000094        |
| 3.78 | .014527 | .000146        | 4.23 | .009264 | .000093        |
| 3.79 | .014382 | .000145        | 4.24 | .009172 | .000092        |
| 3.80 | .014239 | .000143        | 4.25 | .009080 | .000092        |
| 3.81 | .014098 | .000141        | 4.26 | .008990 | .000090        |
| 3.82 | .013957 | .000141        | 4.27 | .008901 | .000089        |
| 3.83 | .013819 | .000138        | 4.28 | .008812 | .000089        |
| 3.84 | .013681 | .000138        | 4.29 | .008724 | .000088        |
| 3.85 | .013545 | .000136        | 4.30 | .008638 | .000086        |
| 3.86 | .013410 | .000135        | 4.31 | .008552 | .000086        |
| 3.87 | .013277 | .000133        | 4.32 | .008467 | .000085        |
| 3.88 | .013145 | .000132        | 4.33 | .008382 | .000085        |
| 3.89 | .013014 | .000131        | 4.34 | .008299 | .000083        |
| 3.90 | .012885 | .000129        | 4.35 | .008216 | .000083        |
| 3.91 | .012757 | .000128        | 4.36 | .008135 | .000081        |
| 3.92 | .012630 | .000127        | 4.37 | .008054 | .000081        |
| 3.93 | .012504 | .000126        | 4.38 | .007974 | .000080        |
| 3.94 | .012380 | .000124        | 4.39 | .007894 | .000080        |
| 3.95 | .012256 | .000124        | 4.40 | .007816 | .000078        |
| 3.96 | .012135 | .000121        | 4.41 | .007738 | .000078        |
| 3.97 | .012014 | .000121        | 4.42 | .007661 | .000077        |
| 3.98 | .011894 | .000120        | 4.43 | .007585 | .000076        |
| 3.99 | .011776 | .000118        | 4.44 | .007509 | .000076        |
| 4.00 | .011659 | .000117        | 4.45 | .007435 | .000074        |
| 4.01 | .011543 | .000116        | 4.46 | .007361 | .000074        |
| 4.02 | .011428 | .000115        | 4.47 | .007287 | .000072        |
| 4.03 | .011314 | .000114        | 4.48 | .007215 | .000072        |
| 4.04 | .011202 | .000112        | 4.49 | .007143 | .000072        |

TABLE V. *Continued*

| $t$  | $A(-t)$ | $\Delta A(-t)$ | $t$  | $A(-t)$ | $\Delta A(-t)$ |
|------|---------|----------------|------|---------|----------------|
| 4.50 | .007072 | .000071        | 4.95 | .004509 | .000046        |
| 4.51 | .007002 | .000070        | 4.96 | .004464 | .000045        |
| 4.52 | .006932 | .000070        | 4.97 | .004420 | .000044        |
| 4.53 | .006863 | .000069        | 4.98 | .004376 | .000044        |
| 4.54 | .006795 | .000068        | 4.99 | .004332 | .000044        |
|      |         |                | 5.00 | .004289 | .000043        |
| 4.55 | .006727 | .000068        |      |         |                |
| 4.56 | .006660 | .000067        | 5.1  | .003881 | .000408        |
| 4.57 | .006594 | .000066        | 5.2  | .003512 | .000369        |
| 4.58 | .006528 | .000066        | 5.3  | .003178 | .000334        |
| 4.59 | .006463 | .000065        | 5.4  | .002875 | .000303        |
|      |         |                | 5.5  | .002602 | .000273        |
| 4.60 | .006399 | .000064        |      |         |                |
| 4.61 | .006335 | .000064        | 5.6  | .002354 | .000248        |
| 4.62 | .006272 | .000063        | 5.7  | .002130 | .000224        |
| 4.63 | .006210 | .000062        | 5.8  | .001927 | .000203        |
| 4.64 | .006148 | .000062        | 5.9  | .001744 | .000183        |
|      |         |                | 6.0  | .001578 | .000166        |
| 4.65 | .006087 | .000061        |      |         |                |
| 4.66 | .006026 | .000061        | 6.1  | .001428 | .000150        |
| 4.67 | .005966 | .000060        | 6.2  | .001292 | .000136        |
| 4.68 | .005907 | .000059        | 6.3  | .001169 | .000123        |
| 4.69 | .005848 | .000059        | 6.4  | .001058 | .000111        |
|      |         |                | 6.5  | .000957 | .000101        |
| 4.70 | .005790 | .000058        |      |         |                |
| 4.71 | .005733 | .000057        | 6.6  | .000866 | .000091        |
| 4.72 | .005676 | .000057        | 6.7  | .000784 | .000082        |
| 4.73 | .005619 | .000057        | 6.8  | .000709 | .000075        |
| 4.74 | .005563 | .000056        | 6.9  | .000642 | .000067        |
|      |         |                | 7.0  | .000581 | .000061        |
| 4.75 | .005508 | .000055        |      |         |                |
| 4.76 | .005453 | .000055        | 7.1  | .000525 | .000056        |
| 4.77 | .005399 | .000054        | 7.2  | .000475 | .000050        |
| 4.78 | .005345 | .000054        | 7.3  | .000430 | .000045        |
| 4.79 | .005292 | .000053        | 7.4  | .000389 | .000041        |
|      |         |                | 7.5  | .000352 | .000037        |
| 4.80 | .005239 | .000053        |      |         |                |
| 4.81 | .005187 | .000052        | 7.6  | .000319 | .000033        |
| 4.82 | .005135 | .000052        | 7.7  | .000288 | .000031        |
| 4.83 | .005084 | .000051        | 7.8  | .000261 | .000027        |
| 4.84 | .005034 | .000050        | 7.9  | .000236 | .000025        |
|      |         |                | 8.0  | .000214 | .000022        |
| 4.85 | .004984 | .000050        |      |         |                |
| 4.86 | .004934 | .000050        | 8.1  | .000193 | .000021        |
| 4.87 | .004885 | .000049        | 8.2  | .000175 | .000018        |
| 4.88 | .004836 | .000049        | 8.3  | .000158 | .000017        |
| 4.89 | .004788 | .000048        | 8.4  | .000143 | .000015        |
|      |         |                | 8.5  | .000130 | .000013        |
| 4.90 | .004740 | .000048        |      |         |                |
| 4.91 | .004693 | .000047        | 8.6  | .000117 | .000013        |
| 4.92 | .004647 | .000046        | 8.7  | .000106 | .000011        |
| 4.93 | .004601 | .000046        | 8.8  | .000096 | .000010        |
| 4.94 | .004555 | .000046        | 8.9  | .000087 | .000009        |
|      |         |                | 9.0  | .000079 | .000008        |

TABLE V. *Continued*

| $t$  | $A(-t)$ | $\Delta A(-t)$ | $t$  | $A(-t)$ | $\Delta A(-t)$ |
|------|---------|----------------|------|---------|----------------|
| 9.1  | .000071 | .000008        | 11.6 | .000006 | .000000        |
| 9.2  | .000064 | .000007        | 11.7 | .000005 | .000001        |
| 9.3  | .000058 | .000006        | 11.8 | .000005 | .000000        |
| 9.4  | .000053 | .000005        | 11.9 | .000004 | .000001        |
| 9.5  | .000048 | .000005        | 12.0 | .000004 | .000000        |
| 9.6  | .000043 | .000005        | 12.1 | .000004 | .000000        |
| 9.7  | .000039 | .000004        | 12.2 | .000003 | .000001        |
| 9.8  | .000035 | .000004        | 12.3 | .000003 | .000000        |
| 9.9  | .000032 | .000003        | 12.4 | .000003 | .000000        |
| 10.0 | .000029 | .000003        | 12.5 | .000002 | .000001        |
| 10.1 | .000026 | .000003        | 12.6 | .000002 | .000000        |
| 10.2 | .000023 | .000003        | 12.7 | .000002 | .000000        |
| 10.3 | .000021 | .000002        | 12.8 | .000002 | .000000        |
| 10.4 | .000019 | .000002        | 12.9 | .000002 | .000000        |
| 10.5 | .000017 | .000002        | 13.0 | .000001 | .000001        |
| 10.6 | .000015 | .000002        | 13.1 | .000001 | .000000        |
| 10.7 | .000014 | .000001        | 13.2 | .000001 | .000000        |
| 10.8 | .000013 | .000001        | 13.3 | .000001 | .000000        |
| 10.9 | .000012 | .000001        | 13.4 | .000001 | .000000        |
| 11.0 | .000011 | .000001        | 13.5 | .000001 | .000000        |
| 11.1 | .000010 | .000001        | 13.6 | .000001 | .000000        |
| 11.2 | .000009 | .000001        | 13.7 | .000001 | .000000        |
| 11.3 | .000008 | .000001        | 13.8 | .000001 | .000000        |
| 11.4 | .000007 | .000001        | 13.9 | .000001 | .000000        |
| 11.5 | .000006 | .000001        | 14.0 | .000001 | .000000        |



## BIBLIOGRAPHY

### BIOMETRIKA—Editorials:

1. *On an Elementary Proof of Sheppard's Formulæ for Correcting Raw Moments and on Other Allied Points.* BIOMETRIKA, vol. III. 1904.
2. *On the Probable Errors of Frequency Constants.* BIOMETRIKA, vol. II. 1902-03.

### BYERLY:

*An Elementary Treatise on Fourier's Series and Spherical Harmonics.* Boston. 1903.

### CARSLAW:

*Introduction to the Theory of Fourier's Series and Integrals.* New York. 1906.

### CARVER:

*The Mathematical Representation of Frequency Distributions.* QUARTERLY PUBLICATION OF THE AMERICAN STATISTICAL ASSOCIATION, vol. XVII. 1921.

### CHARLIER:

1. *Über das Fehlergesetz.*
2. *Die Zweite Form des Fehlergesetz.*
3. *Über die Darstellung willkürlicher Funktionen.*
4. *Weiter über das Fehlergesetz.*

MEDDELANDEN FRÖN LUNDS ASTRONOMISKA OBSERVATORIUM  
vol. II. 1904-09.

### EDGEWORTH:

*The Law of Error.* CAMBRIDGE PHILOSOPHICAL TRANSACTIONS, vol. XX. 1905-09.

### ELDERTON:

*Frequency Curves and Correlation.* London. 1906.

### FISHER:

*The Mathematical Theory of Probabilities*, vol. I. New York. 1922.

### HOBSON:

*The Theory of Functions of a Real Variable.* Cambridge University Press. 1907.

### JØRGENSEN:

*Undersøgelser over Frequensflader og Korrelation.* Copenhagen. 1916.

### KAPTEYN:

*Skew Frequency Curves in Biology and Statistics.* Groningen. 1903.

### PEARSON:

*On the Systematic Fitting of Curves to Observations and Measurements.* BIOMETRIKA, vol. I and vol. II. 1901-03.

### THIELE:

*Theory of Observations.* London. 1903.

VAN UVEN:

*Skew Frequency Curves.* AMSTERDAM AKADEMIE VAN WETENSCHAPPEN PROCEEDINGS, vol. XIX<sup>1</sup>.

WICKSELL:

*On the Genetic Theory of Frequency.* ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK, vol. XII. 1917.

# MISCELLANEOUS MATHEMATICAL TABLES

BECKER AND VAN ORSTRAND:

*Smithsonian Mathematical Tables—Hyperbolic Functions* (Second Reprint), Washington, D. C., 1909.

GLOVER:

*Tables of Applied Mathematics in Finance, Insurance, Statistics,* Ann Arbor, 1923.

GUDERMANN:

*Potenzial Functionen.* CRELLE'S JOURNAL FÜR DIE REINE UND ANGEWANDTE MATHEMATIK, vols. 8 and 9. (This contains tables of logarithms of hyperbolic functions to nine significant figures.)

DE HAAN:

*Nouvelles Tables D'intégrales Définies,* Leide, 1867.

PEARSON:

*Tables for Statisticians and Biometricians,* Cambridge University Press, 1914.

PEIRCE:

*A Short Table of Integrals* (Second Revised Edition), Boston.

PERNOT AND WOODS:

*Logarithms of Hyperbolic Functions to Twelve Significant Figures,* University of California Press, 1918.







